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Octonions and their application

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Motivation

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"There are exactly four normed division algebras: the real numbers, complex numbers, quaternions, and octonions.

The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete.

The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings.

But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative." (Prof. Dr. John Baez)

(https://maths-people.anu.edu.au/~the/seminar/abstracts/2012-03-15-Huerta.pdf 03.11.2019)

Contents

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1. Introduction

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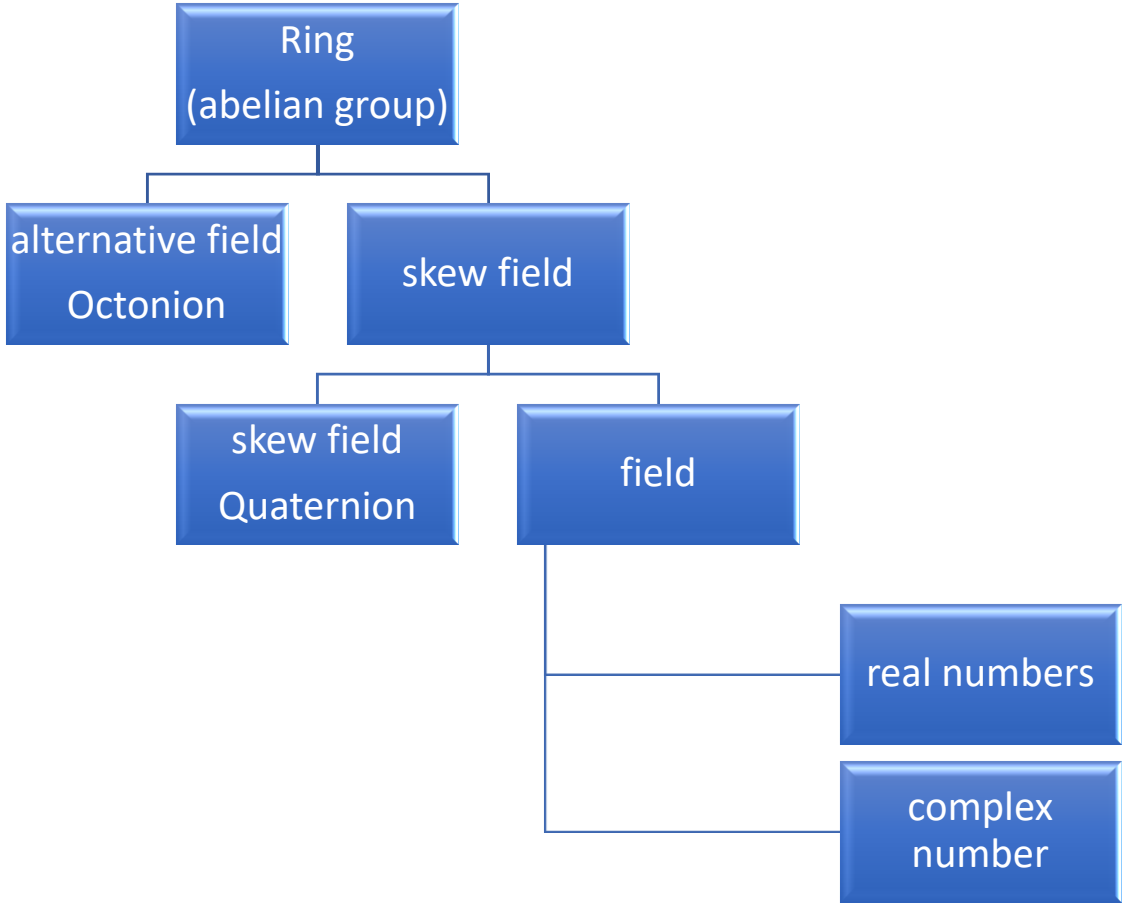
Octonions

- eight-dimensional number system
- alternative algebra (-> multiplication is not associative)
- first mentioned in 1843 by John Thomas Graves
- officially published in 1845 by Arthur Cayley
- Cayley Numbers
- represented: $\mathbb{O}$

2. Octonions- a number system

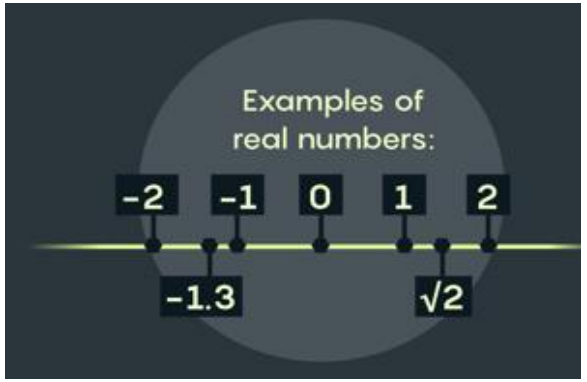
Excurs- hierarchy of algebraic structure



The axioms of a group are deposited in the appendix

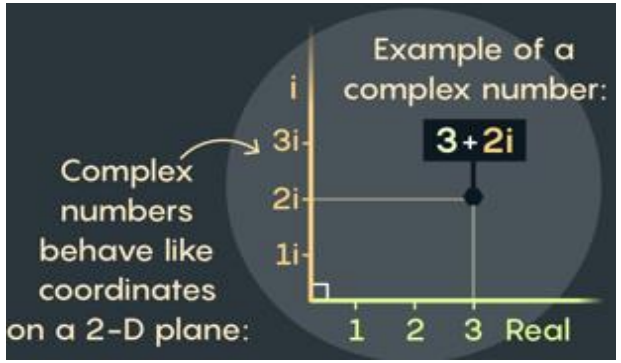
2.1 From the Real Numbers to the Octonions

Real Numbers $\mathbb{R}$



- definition range: -∞ to +∞
- +, -, *, /
- axiom (1) to (5) valid

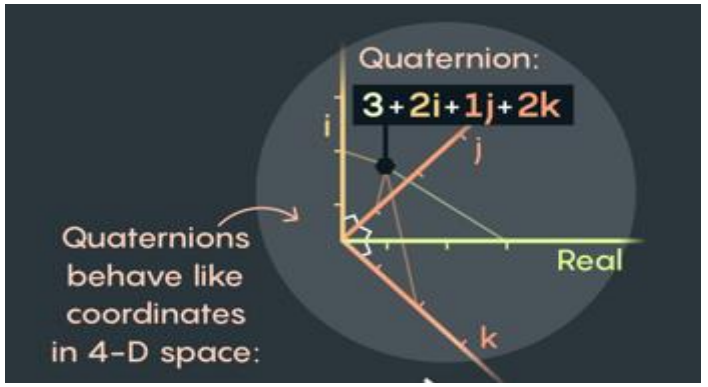
Complex Numbers $\mathbb{C}$



- composed of the real numbers and an imaginary number
- **C** = $a + \mathrm{b} * \mathrm{i}$
- $\mathrm{i}^2 = -1$
- $\forall\, \mathrm{a}, \mathrm{b} \in \mathbf{R}$ and $\mathrm{i} \in \mathbf{C}$
- axioms (1) to (5) are valid
- rotation in the plane
- important for the quantum mechanics

2.1 From the Real Numbers to the Octonions

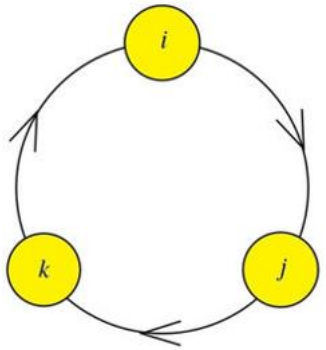
Quaternions $\mathbb{H}$



- axioms (1) to (4) are valid
- skew field (-> noncommutative)
- one reel axes and three imaginary axis k,i,j

$$\mathbf{H} = a + bi + cj + dk$$
$$k^2 = -1$$
$$i^2 = -1$$
$$j^2 = -1$$

$\forall\ a,b,c,d \in \mathbf{R}\ ,\ i,j,k \in \mathbf{C}$



- noncommutative

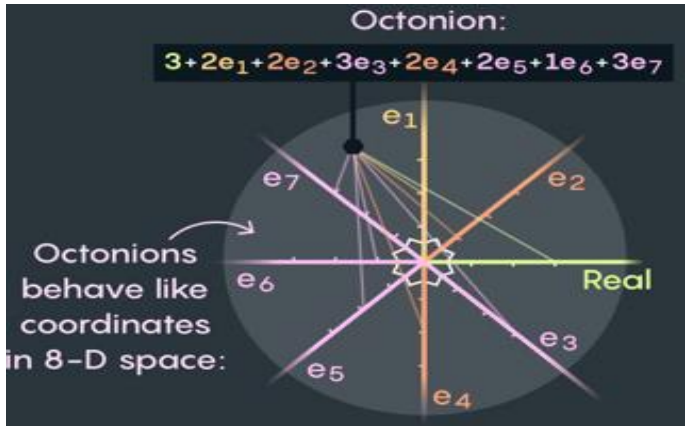
$$i * j = -j * i = k$$
$$j * k = -k * j = i$$
$$k * i = -i * k = j$$

- for Special Relativity theory by Einstein or in quantum mechanics

2.1 From the Real Numbers to the Octonions

Octanions 𝕆

- 8-dimensional space
- alternative field
- nonassociative
- one reel axis and seven imaginary axes



$\boldsymbol{O} = s + te_1 + ue_2 + ve_3 + we_4 + xe_5 + ye_6 + ze_7$

$\forall s, t, u, v, w, x, y, z \in \boldsymbol{R}$

$e_1, e_2, e_3, e_4, e_5, e_6, e_7 \in \mathbf{C}$

ℂ

2.2 Octonions- algebra

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- represented as a set of quaternion $\mathbb{O} := \{(a, b);\ a, b \in H\}$
- single element and a unique inverse
- division algebra
- not a skew field anymore
  - product of octonions is nonassociative (example Fano-plane)
  - noncommutative (-> quaternion)
- alternative law for multiplication
  - alternative field without associative law and commutative law

2.2 Octonions- algebra

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- axioms alternative field - Octonions
  - For a set M, two binary operations + und *
    - operation * is the alternative identity valid

$$l * (l * m) = (l * l) * m \quad \text{and} \quad (l * m) * l = m * (l * l) \qquad \forall\, l, m, n \in \mathbb{O}$$

  - (M,+) , abelian group, neutral element as 0
  - distributive law

$$l * (m + n) = l * m + l * n \quad \text{and} \quad (l + m) * n = l * n + m * n$$

2.2 Octonions- algebra

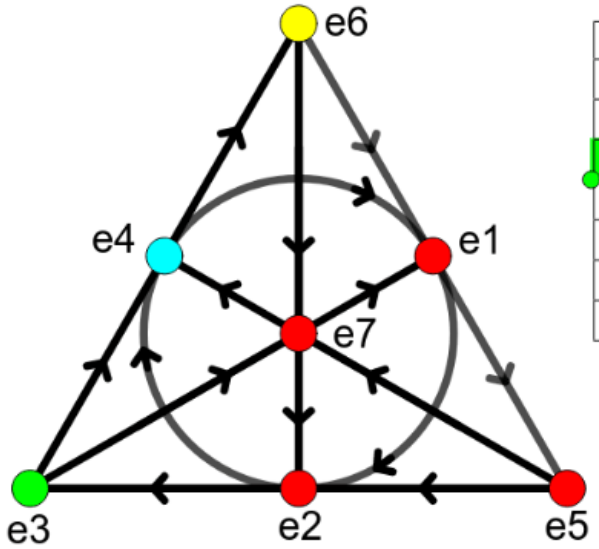
Fano plane

- seven unconventional units en (n=1..7)
- mutliplication in Fano plane

For example, octonions are not associative:

(e5*e2)*e4=e3*e4=e6

e5*(e2*e4)=e5*e1= -e6



	e1	e2	e3	e4	e5	e6	e7
e1	−1	e4	e7	−e2	e6	−e5	−e3
e2	−e4	−1	e5	e1	−e3	e7	−e6
e3	−e7	−e5	−1	e6	e2	−e4	e1
e4	e2	−e1	−e6	−1	e7	e3	−e5
e5	−e6	e3	−e2	−e7	−1	e1	e4
e6	e5	−e7	e4	−e3	−e1	−1	e2
e7	e3	e6	−e1	e5	−e4	−e2	−1

https://www.geogebra.org/m/u35hf2jv

3.Octonions and their application in the physics

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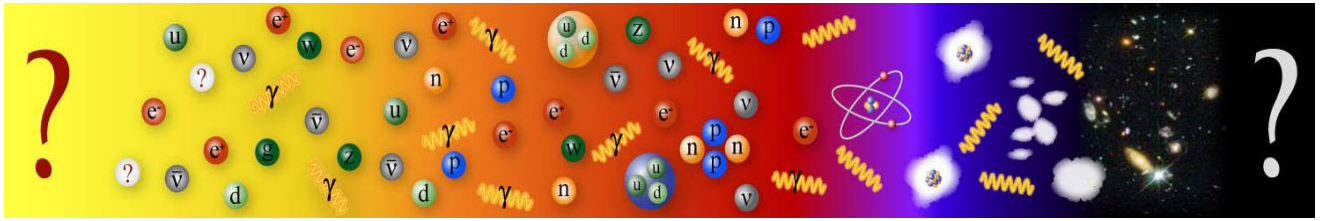
- Octonions be forgotten for a long time
- Octonions could be the key to fully explaining the fundamental forces and particles
- goal of physical research
  - describe matter completely
  - special branch of physics -> particle physic
- the smaller the particles become, the more dimensions are needed



https://www.teilchenphysik.de/

3.1 Standard Modell of Particle Physics

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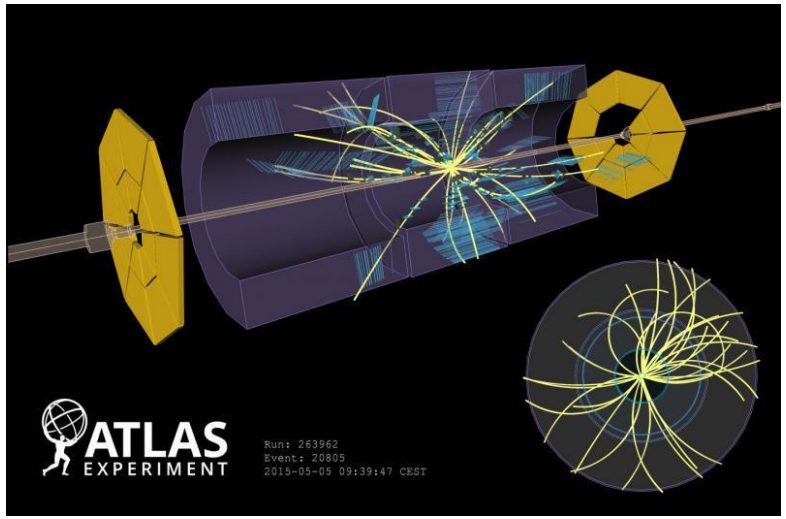
- model to explain the world formula on a very small scale
- still valid and succesfully since 1972
- after the breakthrough of quantum mechanics as it needed as a foundation
- essentially model
  - describe fundamental particles and their interactions
    - building blocks for the universe
- establish in particle physics theory
- experiments are carried out by particle accelerators
  - decayed particles and their energy are measured
- still incomplete



https://www.heise.de/newsticker/meldung/Teilchenbeschleuniger-LHC-Erstmals-wieder-Kollisionen-ausgeloest-2633442.html

3.1.1 Introduction to the Standard Model

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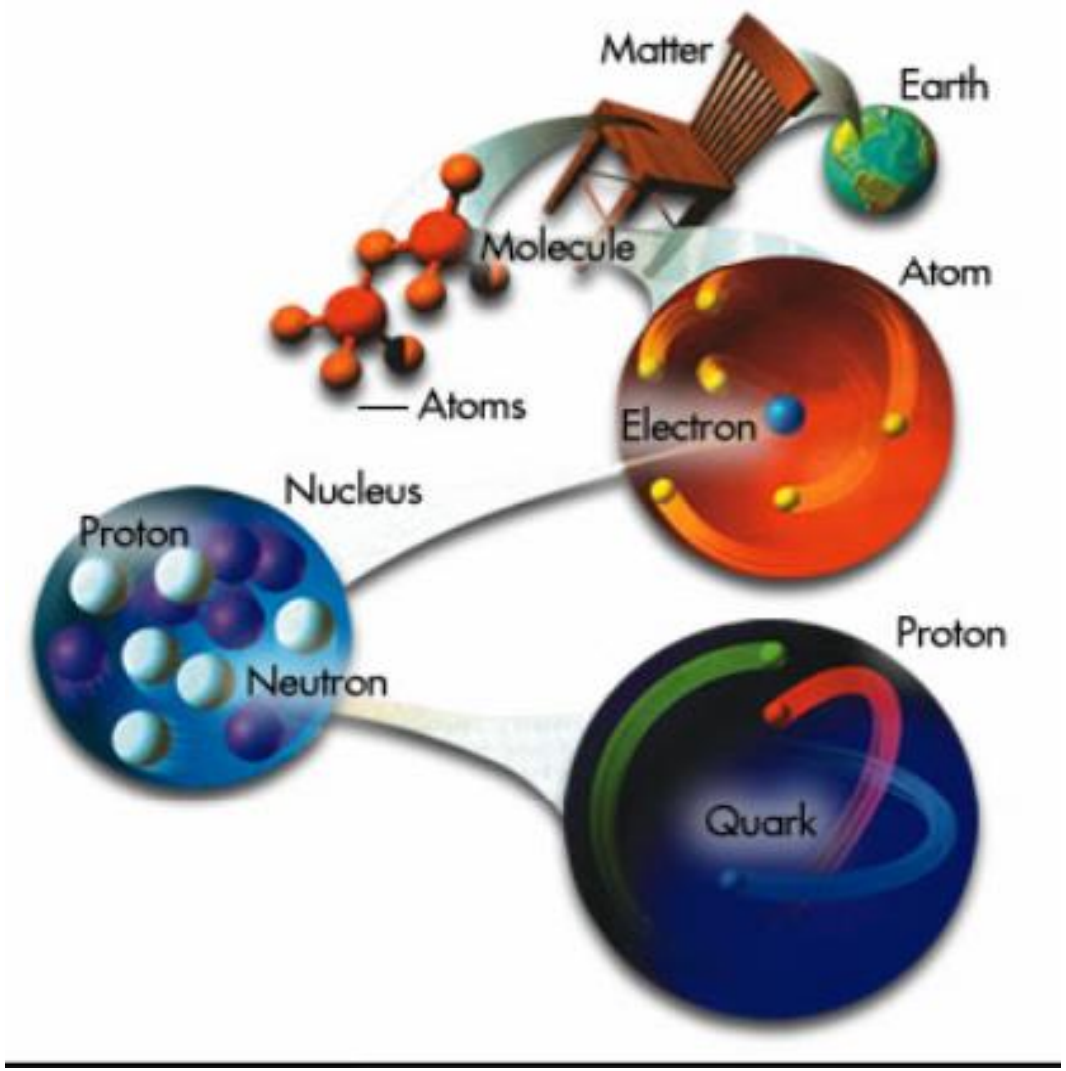
- everything around us is matter
- fundamental particle
  - 6 Quarks
  - 6 Leptons
- electron a fundamental particle
- size:
  - atom (~ 10^{-8} cm)
  - atomic nucleus (~ 10^{-13} cm)
  - proton (~ 10^{-13} cm)
  - electron, quark (~ 10^{-16} cm)



3.1.1 Introduction to the Standard Model

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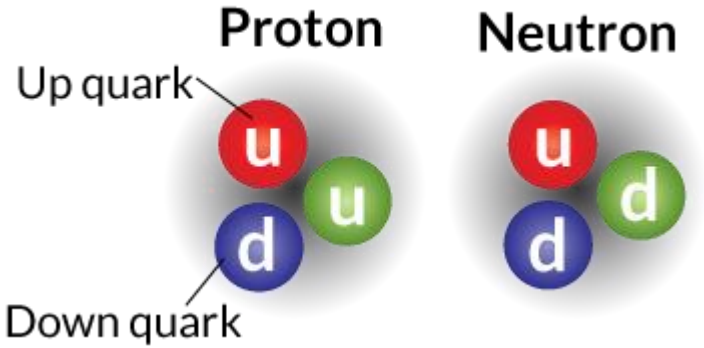
fundamental particles



- Quarks- building blocks of protons and neutrons
- Up-Quark & Down-Quark, strangle-Quark & Charm-Quark, Top-Quark & Bottom-Quark
- each have three color charges: red, blue and green
- Colour charges are different occurrences of the particles

3.1.1 Introduction to the Standard Model

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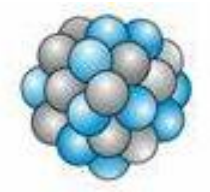
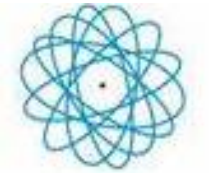
Four fundamental forces



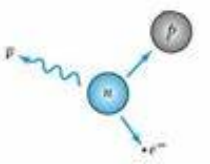
Strong force

- binds the nucleus (protons and neutrons and their building blocks the quarks)



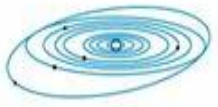
Electromagnetic force

- force binds atoms -> long distance



Weak force

- inside protons and neutrons -> small cross section
- in radioactive decay

()

Gravitation force

- binds the solar system

3.1.1 Introduction to the Standard Model

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Symmetry -> Eichsymmetry

- forces between the elementary particles are determined by symmetries
- internal degrees of freedom of elementary particles
- each force particle has a twin matter particle
- exchange particles against each other

$$SU(3)\ x\ SU(2)\ x\ U(1)$$

- SU(3) - represents the strong power
- SU(2) - represents the weak power
- U(1) - electromagnetic power

Lie group
special unitary group SU(*n*)
unitary group U(*n*)

3.1.2 Octonions as a key in Standard Model- Furey‘s drean

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- Cohl Furey is a mathematician and physics
- researching the relationship of octonions to the Standard Model
- Idea:
  - try to unite all forces (including gravity) mathematically
  - Dixon algebra

$$R \times C \times H \times O$$

  - 1*2*4*8 = 64-dimensional
  - split in two parts by using the symmetry of the particles
    - $R \times C \times H$ -> describes the movement of the elementary particle in four-dimensional space
    - $R \times C \times O$ -> describes the charges of the particles
- to early for results

3.2 Octonions & String Theory- a dream

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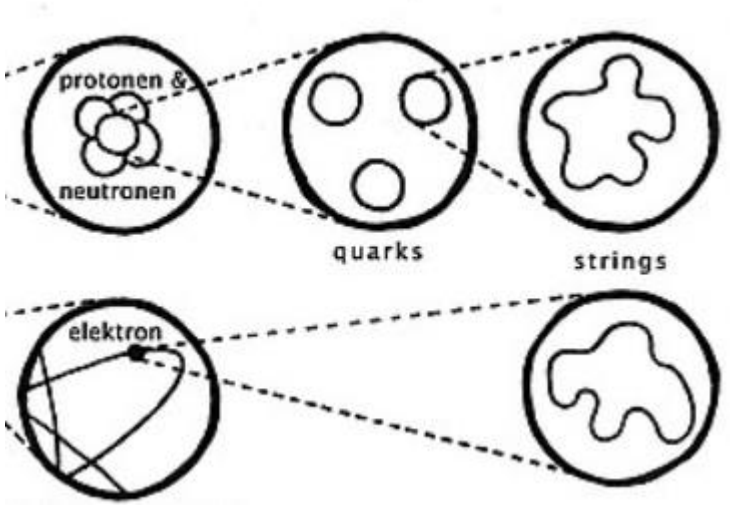
- String Theory offers another model to describe nature under consideration of gravity
- more the one String Theory
- difference to the standard model is the consideration of the particles as string, not as more than one point like particles
- combines theory of relativity and quantum mechanics
- evidence is still pending
- today's accelerator too small for experiments on the theories -> speculative science

3.2.1 Introduction to String Theory

- each elementary particle is described by a string
- open and closed strings (~ 10^{-35} cm)
- strings as a feather chain
  - spring chain that consists of N (= natural number) mass points with the mass m and a strength k
  - N → ∞:
    - m ~ 1/N → 0 , κ ~ N → ∞
  - harmonic oscillator
  - another oscillation produces another particle
  - particles are distinguished by their specific vibrational state
- the Theory of Relativity is still valid



http://www.science-guide.eu/string-theorie.html

3.2.2 Octonions to String Theory

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- Octonions with there eight-dimensions could may explain the universe

  and there 10-dimensionen space time

- as a mathematical basis

- multiplication of octonions, can describe rotation in 8 dimensions

  - particles have a spin (mass and force)

- at the moment there remain all the theories

4. Conclusion

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- Octonions 8-dimensional division algebra
- Octonions- alternative field (-> nonassociative & noncommutative)
- their 8-dimensions could be use to unite all forces in a physic model
- as a key to the world formula
- to early to take theory about the mathematical possibilities and the applications possibilities in the physic

5. References

- Wolchover, Natalie: „Oktonionen Acht Dimensionen für das Standardmodell“ in Spektrum 4.19, 04.2019, S. 66- 73.
- Christoph Berger: Elementarteilchenphysik- Von der Grundlagen zu den modernen Experimenten, 3.Auflage, Springer Spektrum,2014
- https://arxiv.org/pdf/math/0105155.pdf
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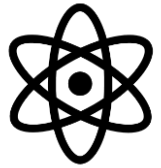

Thanks for your Attention

&

Let‘s talk

5. Algebraic structures for number sets

5.1 Groups

Definition of the structure of a group:

Let G be a nonempty set and $\oplus$ be a binary operation between elements of G.
Then the structure $(G, \oplus)$ is called an **abelian group**, if the following axioms are satisfied:

1) $\forall\ a,b \in G: a \oplus b \in G$ — ***intern operation***

2) $\forall\ a,b,c \in G: (a \oplus b) \oplus c = a \oplus (b \oplus c)$ — ***associative law***

3) $\exists\ e \in G\ \forall\ a \in G: e \oplus a = a \oplus e = a$ — ***identity element***

4) $\forall\ a \in G\ \exists\ a^{-1} \in G: a^{-1} \oplus a = a \oplus a^{-1} = e$ — ***inverse element***

5) $\forall\ a,b \in G: a \oplus b = b \oplus a$ — ***commutative law***

only axiom 1):	magma (groupoid)
only axioms 1), 2):	semi-group
only axioms 1), 2), 3), 4):	group

Examples: $(\mathbb{Z},+)$ for an infinite group $(\mathbb{Z}_n,+)$ for a finite group