UNIVERSITY OF APPLIED SCIENCES WEDEL

Seminar Paper

Octonions and their applications

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1. Introduction

In mathematics and physics, dimensions are important basics for representing systems. However, everything that goes beyond three dimensions is very quickly seen by people as abstract and hardly imaginable. This is exactly where it starts to become interesting for many mathematicians and physicists. Even more exciting is the idea of an eight-dimensional number. The present paper therefore focuses on more dimensions, more precisely on the eight-dimensional number system called octonions and its applications in physics.

In the first part of the paper the octonions are classified and defined in algebra and their properties are discussed. In particular, the closer relatives of the octonions are shown in order to gain a better understanding from the one-dimensional to the eight-dimensional. The second part of the thesis is the application physics.

For a long time, the octonions have been forgotten. One reason for this was that apart from mathematicians, scientists could not do anything with the eight-dimensional numbers. Only with the revolution in physics, more precisely with the breakthrough of quantum mechanics, the forgotten numbers came back into the conversation. The numbers seem to be particularly relevant for the ever-smaller consideration of the particles and thus for the greater use of more dimensions. So, over the last few decades octonions have attained a conceivably important position in particle physics. These extraordinary numbers are considered as the possible key of the world formula to describe the universe. Two great models of modern physics became aware of the octonions. In both directions, physicists have been working on integrating octonions in their research. So, octonions could find their place as a mathematical foundation in the Standard model or even in the String theory. For this reason, chapter 3 and chapter 4 briefly explain the theories and show the connection to the octonions.

Octonions were first mentioned in 1843 by John Thomas Graves. Two years later, in 1845, they were officially and independently published by Arthur Cayley. Therefore, the octonions are referred to as the Cayley Numbers. Octonions are represented by capital letter $\mathbf{O}$ or blackboard bold $\mathbb{O}$.

2. Classification of octonions in the number system

Numbers are like letters a part of our lives. In the first years at school we learn to count and calculate with the real numbers. However, there are more than just the real numbers, which also have an important meaning in some professions.
The following chapter is a short overview of the number systems of real numbers (**R**), complex numbers (**C**), quaternions (**H**) and octonions (**O**).
In the following the definition of algebraic structures, which are relevant for the next chapter, are explained in a short excursion.

2.1 Hierarchical classification of octonion in algebra

In order to be able to investigate octonions, a classification in algebra must first take place. For this purpose, the abelian group and a selection of its subgroups are briefly introduced.
A field is a branch of algebra in which axioms can be added, subtracted, multiplied and divided according to certain rules. As can be seen in the figure (figure 1) below, the real numbers and the complex numbers represent the axioms of the field. A field has almost the same properties as a skew field. With a skew field the multiplication is not commutative. Quaternions are assigned to the oblique fields. If, in addition to the commutative law, the associative law is violated, we speak of an alternative field. Octonions are a famous representative for this. Because these algebraic structures have the properties of a ring.

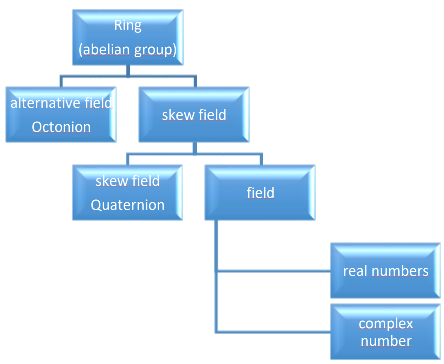


Figure 1: Extract from the algebraic hierarchy (own designed graphics)

Algebraic structures, like other mathematical fields, are subject to definitions, axioms and rules. For octonions and their mathematical relatives, the axioms of the Abelian group are relevant.

A ring is an additively written abelian group. For the abelian group (**G**) axioms there are two binary operations (+, *), addition and multiplication. The following operations are shown for addition but can also be used for multiplication.

Abelian group G consists of an internal two-character link.

intern operation $\forall\ a, b \in \mathbf{G}: a + b \in \mathbf{G}$ (1)

The associative law states that the order in which they are linked is irrelevant.

associative law $\forall\ a, b, c \in \mathbf{G}: (a + b) + c = a + (b + c)$ (2)

The link with the neutral element allows the element to map back itself.

identity element $\exists\ e \in \mathbf{G}\ \forall\ a \in \mathbf{G}: e + a = a + e = a$ (3)

There is an inverse to each element of the group.

inverse element $\forall\ a \in \mathbf{G}\ \exists\ a^{-1} \in \mathbf{G}: a^{-1} + a = a + a^{-1} = e$ (4)

The commutative law says, that two arguments of an operation can be swapped without changing the result.

commutative law $\forall\ a, b \in \mathbf{G}: a + b = b + a$ (5)

2.2 From the Real Numbers to the Octonions

The simplest and most illustrative number system are the real numbers. We usually use these numbers in our everyday life. The real numbers form a field for which the axioms (1) to (5) are valid. We can represent real numbers in one dimension. They are defined in the range from minus infinite to plus infinite. In this plane we can add, subtract, multiply and dived, these properties form the basis for divisional algebra. There are only four numbers systems based on them. This includes the real number, complex numbers, quaternions and octonions.

Real numbers are a point or line (figure 2), because we can lay all the numbers on an infinitely line, like 1, π or minus 4500. Positive numbers can be squared, and the root can be drawn. There are not only positive real numbers, but also negative real numbers. As we know it is not possible to draw the root from negative numbers.

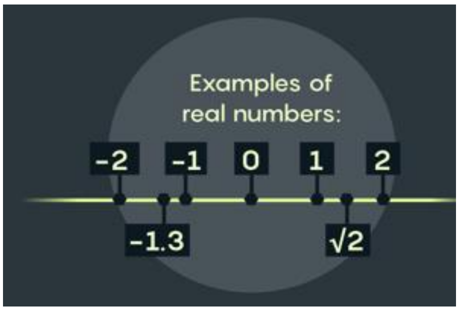


Figure 2: Representation of the real numbers using a number ray (Wolchover, 2019, page 69)

Only with the introduction of the complex numbers these problems could be solved. Complex Numbers are not only indispensable in technology, but also in physics and especially in quantum mechanics. To put in bluntly, quantum mechanics is not possible without the complex numbers. They form a field for which the axioms (1) to (5) are valid. The complex number has a real and a complex part (6). Complex numbers are two dimensional. That means there are two coordinates, one real and one imaginary (figure 3).

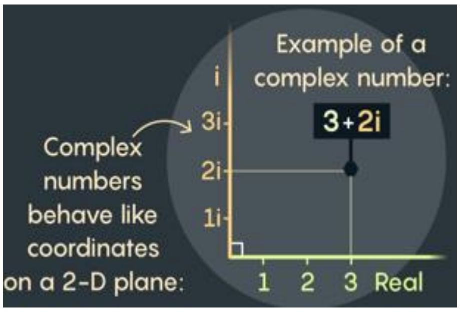


Figure 3: Representation of complex numbers in the coordinate system (Wolchover, 2019, page 69)

What distinguishes the complex numbers from the real numbers is the imaginary number, which in square represents a non-positive real number. (7). The complex numbers have an unconventional imaginary unit called i. The multiplication by i corresponds to a 90-degree rotation in the number plane.

$$\mathbf{C} = a + b * i \tag{6}$$

$$i^2 = -1 \tag{7}$$

$$\forall\, a, b \in \mathbf{R} \text{ and } i \in \mathbf{C}$$

Quaternions with its four dimensions, are the next higher dimension after the complex numbers. The four-dimensional space is represented in the coordinate system (8) by three imaginary axes k (9), I (10) and j (11), and one real axis (figure 4).

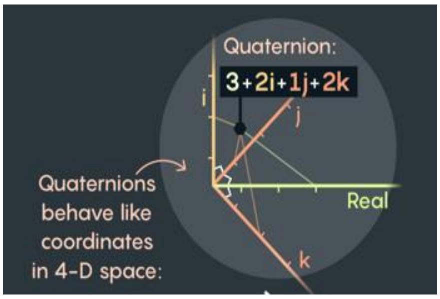


Figure 4: Representation of quaternions in the coordinate system (Wolchover, 2019, page 69)

The following equation (8) provides a possible representation of quaternions.

$$\mathbf{H} = a + bi + cj + dk \qquad \forall\, a, b, c, d \in \mathbf{R},\ i, j, k \in \mathbf{C} \tag{8}$$

While in two dimensions one rotation did not matter, in four dimensions it does. The reason for this is, that the communicative law is violated, that is changing the order of operations which changes the results. We are talking here about a slanting field. That means, that only the axioms (1) to (4) are valid. This deficit of the numbers proved to be an advantage in higher mathematics or even in quantum physics. For example, in the path control of spacecraft or signal processing. Using quaternions as mathematical foundations, the space-time of relativity theory can be represented in quantum mechanics.

Quaternions can also be multiplied like real numbers or complex numbers. The following illustration (figure 5) can be used to illustrate multiplication.

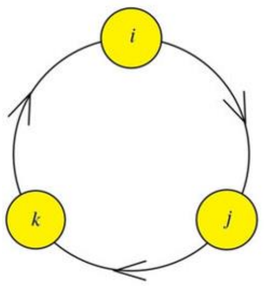


Figure 5: multiplication picture for quaternions by cyclic permutations (References [15])

There are two rules to follow. If two units are multiplied clockwise, the result is positive. An example of this is the multiplication of first I and the j, showing equation (9), which leads to a positive result of unit k.

$$i * j = k \tag{9}$$

However, if two units are multiplied counterclockwise, i.e. first j and then i (as shown in equation 10), the result is negative, in this case it would be -k.

$$j * i = -k \tag{10}$$

This example clearly shows that quaternions are not commutative.

The octonions are an extension of the quaternions in the number system. Octonions are a normed division algebra and represented by the symbol **O**. This numbers can be defined coordinates in an abstract 8-dimensional space. Now we have one real axis and seven imaginary axes (11) (figure 4).

$$\mathbf{O} = s + te1 + ue2 + ve3 + we4 + xe5 + ye6 + ze7 \qquad (11)$$
$$\forall\, s, t, u, v, w, x, y, z \in \mathbf{R} \quad \text{and} \quad e1, e2, e3, e4, e5, e6, e7 \in \mathbf{C}$$

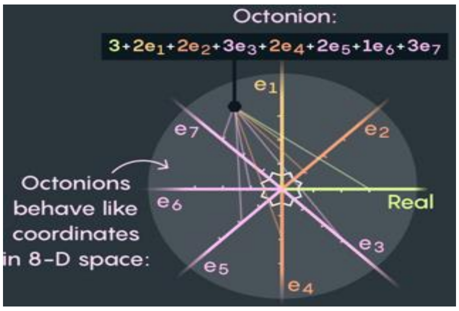


Figure 6: Representation of octonions in the coordinate system (Wolchover, 2019, page 69)

The mathematical definitions and the resulting properties are explained in detail in chapter 2.2.

The next level after the octonions are the Sedenion with their 16-dimensions which do not show any algebraic structure anymore. Their multiplication is neither commutative, associative or alternative.

2.3 Octonions Algebra

The octonions are at the end of numerical term, especially of the normed algebra. This is because octonions do not fulfil two important laws (associative and commutative). One level further on are the sedenions, with their 16-dimensions, they can no longer perform many mathematical operations. Because on their way, they would lose their inverse element. For this reason, the octonions are the end.

The octonions are represented as a set of quaternions (12)[1].

$$\mathbf{O} := \{(a, b);\ a, b \in \mathbf{H}\} \tag{12}$$

They do not form a skew field anymore, because they violate the associative law (2) and the commutative law (5) for the multiplication. Nevertheless, octonions are attributed to the field, namely the attenuated form of the alternative field. For this purpose, the axioms for the octonions are explained below as alternative field.

Furthermore, it is an algebraic structure with a two-character link. There is a neutral element 0 for the addition and 1 for the multiplication.

However, the alternative law is fulfilled for multiplication. Because it fulfils the following axioms of an alternative field. They have alternativity as a property. This is an attenuated form of the associative law shown in the equation (13).

$$l * (l * m) = (l * l) * m \quad \text{and} \quad (l * m) * l = m * (l * l) \qquad \forall\, l, m \in \mathbf{O} \tag{13}$$

Both distribution laws (14) are still valid.

$$l * (m + n) = (l * m) + (l * n) \quad \text{and} \quad (l + m) * n = (l * n) + (m * n) \qquad \forall\, l, m \in \mathbf{O} \tag{14}$$

[1] Only the definitions are explained in this paper. The evidence on the definitions is left out.

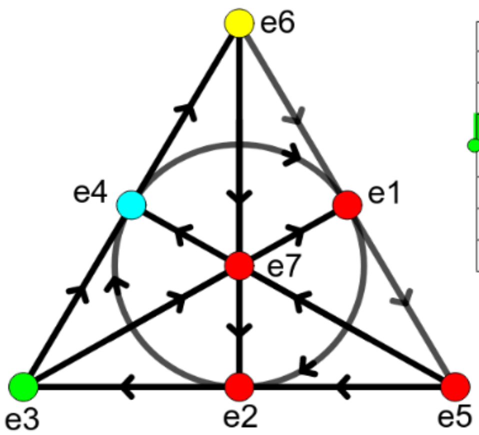
This algebraic structure, which defines a multiplication of the abelian group is called normalized divisional algebra. The distributive set and zero divisor free are fulfilled as well as for a single element exists.

A short digression in this context to the divisional algebra.

Divisional algebra is a branch of abstract algebra. By divisional algebra over real numbers we understand a vector space in which an algebra over a body in vector space represents a representation in the form of an addition or multiplication. Only four finite dimensional divisional algebras on real numbers are known. The number systems considered in this chapter represent the four real, normalized division algebras with the element one. It is especially remarkable that in divisional algebra there are only the dimensions 1,2,4 and 8. So there are only four real division algebras: the real numbers, the complex numbers, the quaternions and the octonions. This applies according to Hurwitz's theorem.

In the previous chapter we already discussed the representation of the octonion in the coordinate system. These numbers have seven unconventional units e1, e2, e3, e4, e5, e6, e7. The multiplication is encoded in the „Fano plane“ (figure 7). This means if you multiply two neighbour elements on a line, the result is the next element on that same line following the arrows. So that closes the loop for each group of three element in an additional line. A multiplication against the direction of the arrow leads to a negative result.

The multiplication matrix (figure 7) can be used to multiply the individual units. Each variant of the multiplication is shown in the matrix.



	e1	e2	e3	e4	e5	e6	e7
e1	−1	e4	e7	−e2	e6	−e5	−e3
e2	−e4	−1	e5	e1	−e3	e7	−e6
e3	−e7	−e5	−1	e6	e2	−e4	e1
e4	e2	−e1	−e6	−1	e7	e3	−e5
e5	−e6	e3	−e2	−e7	−1	e1	e4
e6	e5	−e7	e4	−e3	−e1	−1	e2
e7	e3	e6	−e1	e5	−e4	−e2	−1

Figure 7: Fano Plane and multiplication matrix (References [14])

For example, octonions are not associative:

$$(e5 * e2) * e4 = e3 * e4 = e6 \quad (15)$$

$$e5 * (e2 * e4) = e5 * e1 = -e6 \quad (16)$$

In the example above, e5, e2 and e4 are multiplied in different order. The multiplication leads to two different results, one being e6 (15) and the other -e6 (16). Thus, it was shown that the associative law no longer applies to octonions.

3. Octonions and their application in the physics

In recent years, physics has increasingly recognized the relevance of the strange number system, the octonions. Particle physicists see a special benefit in the octonions, that they could be the key to fully describe the fundamental forces and particles. In order to draw the connection between the octonions and physics, this chapter shows the Standard Model and the String Theory as an area of application.
The great goal of physical research is to describe matter completely, i.e. the structure of matter and its individual basic building blocks. All phenomena in the world and in the universe can be explained by these fundamental forces and interactions. This branch of physics is called particle physics. The structure of elementary particles today is primarily describing by the Standard Model, which will be explained in more detail in the next chapter.

3.1 Standard Model of Particle Physics

The Standard Model is a model of the particle physics which is used to describe particles from which matter is formed. The model is valid since 1972. The associated theories have been supported by many experiments in recent decades. Quantum mechanics and the theory of relativity form the theoretical pillars of the Standard model, which cloud not be established without these theories.
The considered fundamental forces and elementary particles are very small. They can only be measured by experiments at the accelerator. Especially at CERN a lot of research is successfully carried out in the direction of the Standard Model.

3.1.1 Introduction to the Standard Model

In the school ages it is already taught that everything in nature consists of molecules. A molecule consists of atoms and is held together by interactions. The building blocks of molecules is an atom. The size of an atom is ~10^{-8} cm. The components of an atom are the atomic nucleus with ~10^{-13} cm and the electrons, where one electron is ~10^{-16} cm. An atomic nucleus in turn consists of protons and neutrons, which are about ~10^{-13} cm in size.

The Standard Model considers elementary particles, the basic building blocks of protons and neutrons. Elementary particles are fermions, which are divided into six quarks and six leptons. As well as interactions which are called bosons. In addition, there is an anti-particle to each particle which doubles the number of elementary particles. For thematic reasons, only the quarks are explained in more detail in this paper, leaving out the other particles. Quarks are not elementary particles, they exist only in bond states. There are up-Quarks and down-Quarks, strangle- and charm-quarks, as well top- and bottom-quarks. They can each have three colour charges, red, blue and green. Colour charges are nothing else than different occurrences of the particles. The protons have two up quarks and one down quark as basic building block. The neutrons have two down quarks and one up quark.

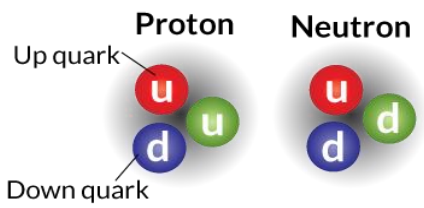


Figure 8: building blocks of Proton and Neutron (References [16])

In physics, force (interactions) is indispensable. Therefore, fundamental forces play an important role. Fundamental because all force effects in nature can be traced back to them. The Standard model considers fundamental forces. The four fundamental forces are the strong force, the weak force, the electromagnetic force and the gravitational force.

When we talk about the strong power, we usually mean nuclear power. This exerts a strong interaction among the particles (protons and neutrons and their building block the quarks). The strong force binds the quarks.
The weak force acts inside the protons and neutrons. Since it acts at the smallest distance and has a small cross section of interaction, it is called a weak force. This weak force acts as an interaction between the neutrons and is also responsible for radioactive decay of atomic forces.
The electromagnetic force acts over long distances and holds an atom together. The interaction takes place between differently charged particles. For example, the negatively charged electrons and the positively charged nucleus.
Finally, the gravitational force, which acts over a very large distance and a large amount of matter. Gravity therefore plays an important role in our universe, In the Standard Model only very small particles are considered, with a very small mass, so this force is left out.

Symmetries are basic properties for the Standard Model. Even the forces between the elementary particles are determined by symmetries. The deeper reason has not yet been researched. In the Standard Model three symmetry groups are used to describe how to exchange particles (electrons and quarks) for each other. It means that every particle has a partner, a particle which carries the force. This symmetry is called "Eich" symmetry (17). This symmetry is a local and continuous symmetry from the gauge theory, which became important for internal degrees of freedom of elementary particles. Symmetry is particularly important for the interactions.

$$\mathrm{SU}(3) \text{ x } \mathrm{SU}(2) \text{ x } \mathrm{U}(1) \tag{17}$$

SU(3) represents the strong power, SU(2) the weak power and U(1) the electromagnetic power. This symmetry forms the algebraic structure of a group. More specifically to the "Eich" symmetries, cause the parameters assume continuous values.
The "Eich" symmetry is described in quantum mechanics for the wave function. Mathematically, the Lie group U(1) is used for this. Since this is a basis in the equations in the Standard Model, there are researchers who try to include octonions in the equations. The symmetry of the octonions is considered.

The basic theorem of mathematics is used (similar of complex numbers), in which eight independent number are given the length of one. With these numbers, the entire number space is spanned. Thus, symmetries can be recognized with simple mathematical operations. But these are theories that are still in their infancy and have not yet been fully established.

3.1.2 Octonions as a key in Standard Model- Furey's dream

All these forces and particles are represented mathematically in the Standard Model. The Standard Model thus contains all elementarily important forces except for the gravitational force, which is an interaction. In addition, particles still have other properties that cannot yet be explained easily. This is one of the reasons why the detectors at CERN probably will find more particles that the Standard Model does not predict. It is assumed that the octonions could combine all four basic forces. Experiments in elementary particle physics are carried out by collisions at particle accelerators. The detectors then pick up charges, origin, number momentum and energy of the particles. But there are still more questions than answers. Therefore, there are different approaches in physics in which researchers try to combine the octonions and the Standard model. One of them is Cohl Furey.

The mathematician Furey, through the eight dimensionalities of the octonions, tries to determine the mass of the elementary particles, the Higgs mechanism, gravity and space-time. In other words, she tries to mathematically unite all forces including the gravitational force. One approach among many is the attempt to use divisional algebra. The dimensions could then be described by one electron, one neutrino, three down quarks and three up quarks. Furey goes one step further and incorporates the Dixon algebra (18) into the theory of describing the Standard model.

$$R \times C \times H \times O \tag{18}$$

If the four spaces are multiplied by one another, i.e. 1*2*4*8, there is a 64-dimensional space obtained. Furey describes the elementary particles in a subspace that is never left. Thus, a particle regardless of whether it moves or interacts with another particle. It splits the Dixon algebra into two parts.

First the quaternion part. This part ($R \times C \times H$) describes the movement of the elementary particle in four- dimensional space. The special theory of relativity is thus considered. The second part describes the charges of the particles ($R \times C \times O$). So, the octonions part could result the symmetry groups of the Standard model (17).

3.2 Octonions & String Theory – a Dream

The Standard model researches today already describe nature very well. Nevertheless, many questions are still unanswered or not considered. For example, the Standard Model neglects gravity. The String Theory offers another model to describe nature under consideration of gravity. For this reason, continuing theories are established to investigate the elementary building block of nature. One model is called String theory. In this model, the particles are no longer considered as points but as strings. In this chapter you will find a short introduction to String theory and which role octonions could play in it. There are countless theoretical approaches to String theory. In this paper only the basic features of the model are considered for reasons of simplifications.

3.2.1 Introduction to String Theory

String theory is a collection of models and a part of particle physics. In this thesis only the basic properties are explained, the different expressions and deepings are left out. The goal is to create a model that considers all fundamental forces, including gravity in a quantum mechanical construct. String theory combines the theory of relativity and of quantum mechanics. Compared to the Standard model, spacetime is integrated in this model.

In the previous chapter, the Standard model was explained. The Standard model examines elementary particles in more detail. In String theory each elementary particle is described by a string. There are different forms of strings. They are often considered as a closed or an open chain. One string is about 10^{-35} cm small. The string are assumed to be components of the quarks.

Elementary particles are distinguished by their specific vibrational state. In summary this means a unification of the different elementary particles, since only one type is required.
In a very simple classical mechanical way, you can imagine the strings as a feather chain. A spring chain that consists of N (= natural number) mass points with the mass m and a strength k. If we let N run against infinity the mass becomes relative to 1/N which aims towards 0 and the strength proportional to N against infinity. N springs coupled together reflect a harmonic oscillator. If the harmonic oscillator is quantized the result is a quantized string. This will not be further discussed in this paper as it requires the basic of quantum mechanics.
String theory cannot be tested experimentally now, it is still speculative science. The high mathematical complexity makes it difficult to draw up concrete calculations. Nevertheless, it remains a very cutting field with many degrees of freedom for further theories.

3.2.2 Octonions in the String Theory

There are theories that the strangest numbers called Octonions may explain the universe and the 10-dimensional space time. The String theory is in the present time, the theory which could solve the world formula the closest. The six extra dimensions which are described in String theory cannot be found in our everyday life because they are too small to be perceived.
To explain the String theory, we need a lot more than the three 3 dimensions. The octonions are very close for the mathematics to explain the String theory. For this reason, many String theories try to include the octonions. A precise application in theory does not yet exist. Now there are many approaches that still need time before they can be officially established as theories.

4. Conclusion

The brilliance of the octonions is reflected not only by the different mathematical areas, but also in their special relation to physics.
Octonions are special with their eight dimensions, their alternative laws and their variety of possible applications. Even though they are different in their own way from the other numbers, they have strong connections to the real numbers, complex numbers and quaternions. Exactly these numbers and only with these numbers it is possible to work in divisional algebra.
The great goal in physics is the world formula, which integrates all laws of nature, to completely set up and solve it. Octonions could play an important role in the constellations by the mathematical provability. But experts share the opinion that in the moment it is too early to take theory about the mathematical possibilities and the applications possibilities in the physic.
Even though the Standard Model is still valid since the 1970s and has been repeatedly confirmed in many experiments. It is still not complete. Many questions and parameters remain open. One important fact is, among others, the non-observance of gravity. Could gravity still play an important role in fundamental particles? And what if the particles are not point-like after all but have a string shape or even a completely different shape. Perhaps the octonions can help to answer these questions and more. Now, research in this direction is in the children's shoes. First, further theories would have to be established. And what is even more important, experiments would have to be developed to test the theories. Now the accelerators are too small in diameter to carry out such experiments. But maybe in a few years or decades new concepts will be developed to test exactly the theories, like String theory and the octonions as mathematical basis.

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