Small Worlds and Their Modelling

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Table of Contents

1 Introduction

In our daily lives we are surrounded by networks from the internet and computer networks, to social networks, to the genetic networks in a biological cell. The properties and a deeper understanding of these networks can be used to explain the behaviour and performance of diverse economic, social, technical and biological systems.
In the following paper I'll have a look at the so called "Small World Networks" which are characterized by the fact that the shortest path between two nodes in the network is relatively small compared to their total size. How can small worlds achieve those short paths between the nodes in the network and which mathematical concepts can they be explained and described with?

2 Six Degrees of Separation

In this first part I'm having a closer look at the origin of the the famous "Six Degrees of Separation" in our society and why our society is actually a small world. I'm also looking at other examples for small worlds and how they achieve their low degree of separation.
One of the first and most known experiments to show the evidence of small world features in society is the "Chain Letter Experiment" by Stanley Milgram.

2.1 Stanley Milgram's Chain Letter Experiment

In the late 60s the American social psychologist Stanley Milgram asked himself:

> *"How many acquaintances do you need to connect any two persons in the United States with each other? "*

The degree of separation between two randomly picked persons would be one if they know each other or two if they share a common acquaintance or a friend. The degree of separation is called the least amount of links between two nodes in a network. To find an answer to this question and to be able to measure the "distance" between two randomly picked persons he started his very well-known chain letter experiment. [1]

He choose two target persons: The wife of a graduate student in Sharon, Massachusetts and a stock broker in Boston. The starting points of the Study where Witcha in Kansas and Omaha in Nebraska. Milgram sent letters with the following instructions to randomly chosen residents asking them to participate in a study of social contact in American society[2]:

HOW TO TAKE PART IN THIS STUDY

1. ADD YOUR NAME TO THE ROSTER AT THE BOTTOM OF THIS SHEET, so that the next person who receives this letter will know who it came from.
2. DETACH ONE POSTCARD. FILL IT OUT AND RETURN IT TO HARVARD UNIVERSITY. No stamp is needed. The postcard is very important. It allows us to keep track of the progress of the folder as it moves towards the target person.
3. IF YOU KNOW THE TARGET PERSON ON A PERSONAL BASIS, MAIL THIS FOLDER DIRECTLY TO HIM (HER). Do this only if you have previously met the target person and know each other on a first name basis.
4. IF YOU DO NOT KNOW THE TARGET PERSON ON A PERSONAL BASIS, DO NOT TRY TO CONTACT HIM DIRECTLY. INSTEAD, MAIL THIS FOLDER (POSTCARDS AND ALL) TO A PERSONAL ACQUAINTANCE WHO IS MORE LIKELY THAN YOU TO KNOW THE TARGET PERSON. You may send the folder to a friend, relative or a acquaintance, but it must be someone you know at a first name basis. [3]

In the end 42 of the 160 letters sent by Milgram made it to one of the target persons although some required close to a dozen intermediates. He found out that the median number of intermediate persons was 5.5. Which is a relatively small number compared to the population in the US, which was 198.7 million people in 1967. [4][5]
Due to the fact that the people who participated in the experiment “just guessed” a person who might know the target person means that there could even exists a shorter and more optimal path to the target person which they didn’t know of. Consequently, the existence of a chain letter of the length k between a starting person and a target person shows that the shortest path between them has maximal the length k. It could be shorter than k, because the chain letter is not always taking the optimal path. [6]

There are critical voices to Stanley Milgram’s experiment, as well. The psychologist Judith Kleinfeld points out that the amount and the social background of the test persons are not representative enough.[7]

Nevertheless, Stanley Milgram showed with his experiment that we live in a world where no one is more than a few persons separated from each other.
Society is a small world cause it forms a very dense web of social interactions especially today when it was never easier for us to stay in contact despite of long physical distances thanks to the internet. But is this only possible because of the human desire to stay connected or can this be applied to other scientific fields, as well? Are other networks small worlds, too?

2.2 Other Small Worlds and Degrees of Separation

The small world phenomenon caught the attention of a lot of scientists in the last couple of years. The network scientist Albert-Laszlo Barabási wanted to know:

> *“How big is the distance between any two documents on the World Wide Web?”*

The World Wide Web is a network whose nodes are the webpages and the links are the uniform resource locators (URLs) which make it possible to get with a click from one web document to the other. These links turn the collection of individual documents and web pages into a huge network. With an estimated size of over 130 trillion documents[8], the Web is one of the largest network humans have ever built[9].

Barabasi and his team used a software called a crawler to map out the Web’s connections. A crawler can start from any webpage, identifying the links (URLs) on it. In the next step it downloads the documents these links point to and again identifies the links on these documents. This process repetitive returns a local map of the Web.[10]
Search engines like Google or Bing use crawlers to find and index new documents.This is a massive advantage in studying the WWW compared to society which is not digital in total. When they first tried to create a map of the WWW in 1998 the internet had approximately only around 800 million nodes. Albert Laszlo Barabasi and Hawoong Jeong first started to only map the nd.edu domain. They applied their discoveries at the entire WWW and figured

out that the degree of separation is in average 18.59, defined as the smallest number of URLs that must be followed to navigate from one document to the other. In average every randomly chosen document in the web is separated by only 19 clicks.[11] The results of their research indicate that the average separation between the nodes increases more slowly than the number of documents. “If we denote d to be the average separation between the nodes on a Web of N Webpages, the separation follows the equation $d = 0.35 + 2logN$, where $logN$ denotes the base-10-logarithm of N”[12]

Regardless of the fact that the WWW has grown a lot since 1998 from 800 million nodes to 130 trillion because of it’s small world properties the degree of separation of the WWW is not supposed to drastically increase. [13]
The research of Albert Laszlo Barabasi and his team did not only show that the WWW is a small world it also revealed some completely new results about networks which had a massive impact on understanding real networks and how e.g documents can be easily accessed despite of the massive amount of documents out there. These results lead to the development of Scale-Free networks in 1999.[14]

Other networks which show small world features are e.g. species in food webs who have in average a degree of separation of two, molecules in the cell who are separated on average by three chemical reactions and the internet which is separated by routers has a degree of separation of ten.[15]

The WWW’s nineteen degrees of separation may seem very large compared to society with six or the internet with ten. The important fact is that networks with a massive amount of nodes shrink, displaying a separation far shorter than the amount of nodes they have. [16]

2.3 Why is the Degree of Separation so low?

How is it possible e.g. for society to have a degree of separation of six although society consists of seven billion people? The answer to this question is the highly interconnected nature of those networks. To stay connected the critical number would be one link. But in real life we all know more than one person. As soon as we start to add more links the distance decreases.

As an example, in a network where the nodes have on average k links. From one node it is possible to reach k other nodes with one step. This means k^2 nodes are two links away and k^d nodes are d links away. Consequently, if k is large with a few steps it is possible to reach most of the nodes.
This can be turned into a formula that predicts the separation in a random network.
“If we have N nodes in the network k^d must not exceed N. Using $k^d = N$ we get a simple formula which tells us the average separation follows the equation $d = logN/logk$” [17]

If we take society as as example $N = 7\ 000\ 000\ 000$ and $k = 1000$. A thousand acquaintances might sound like a lot but it's actually not if we think about all the people we know. This leads to $d = log(7\ 000\ 000\ 000)/log(1000)$ and $d = 3,282$

This might be a little bit over estimated for society but it's a fact that it was never so easy for us in times of the internet compared to 1967 that scientists like Albert-Laszlo Barabási estimate our degree of separation in society down to 3 today.[18]

3 Ways of Modelling

Over the years there have been different attempts of modelling small world networks. One of the first and very often considered ones are Random Graphs. Moreover, I'll have a closer look at Small World Graphs and Scale Free Networks.

3.1 Random Graphs

Random Graphs were developed by the mathematicians Paul Erdős and Alfréd Rényi in 1959. These graphs have two parameters: n which is the number of nodes in the graph and a positive integer, and p a number between 0 and 1 which is the probability that two nodes are linked with each other. A Random Graph with these parameters is denoted as $G(n, p)$ and can be created like this:

1. The nodeset ($v_1, ..., v_n$) consists of n nodes.
2. Two nodes v_i and v_j with $i < j$ are linked with the probability p. This happens separately for different pairs of nodes. [19]

Let's have a closer look at the behavior of Random Graphs, if p is set and $n \to \infty$ because it shows how the model behaves for very big n.
The questions is: "What is the possibility of two nodes being linked over one common neighbour?" In other words: "How big is the chance that a way of the length two exists?" The nodes v_i and v_j are linked with the probability p. This means that the probability that a way with the length two exists depends on n. Because there are $n - 2$ of those ways (one for each node except v_i and v_j) and the probability that the nodes are in the Random Graph is p^2. Thereof the probability that **none** of these $n - 2$ possible ways is in the Random Graph is:

$$(1 - p^2)^{n-2}$$

This probability for $n \to \infty$ goes towards 0. Consequently this means, that for $n \to \infty$ the probability that a way of the length two exists goes towards 1. To sum it up, almost every pair of nodes shares a common acquaintance which would be too much if we apply this to real networks. [20]

To make sure the amount of direct neighbours doesn't grow too big. It would be possible to say $p = \frac{c}{n}$ where $c > 0$ is a positive constant. The expected amount of links to one node would be $(n - 1)p = \frac{(n-1)c}{n}$ because for $n \to \infty$ $\frac{(n-1)}{n}$ goes towards 1. The expected amount of neighbours for big n is c. [21]

Despite of the links' random placement all the nodes in the graph will have approximately the same amount of links. Compared to society this would mean that we all have roughly the same amount of acquaintances or that for example every web page has the same amount of links to other web pages. Moreover, this implicits that all the nodes in the network have the same chance to be linked to each other. This does sound quite right but a what does this actually mean e.g. for society.

A Random Graph with $G(n, \frac{c}{n})$ with $c = 1000$ and $n = 7\,000\,000\,000$. Every pair of nodes has the chance $\frac{c}{n} = \frac{1}{7\,000\,000}$ to know each other. Under the 1000 acquaintances are $\frac{1000 \times 999}{2} \approx 500\,000$ possibilities to choose two persons. Each of these 500 000 pairs has the probability $\frac{1}{7\,000\,000}$ to know each other. The expected amount of acquaintances who know each other is $\frac{500\,000}{7\,000\,000} \approx 0.071$. This means that the probability that two acquaintances from a person know each other is less than 8 percent. [22]

If we apply this to real life it doesn't seem very convincing that the chances are the same that my best friends know each other as that I know a pizza baker from Venice. In most cases two good friends know each other's friends, because society doesn't work completely random. A group of friends is very likely going the same pubs, gyms or universities.[23] This is why random graphs are not realistic enough to model e.g. real social networks like society.

3.2 Small World Graphs

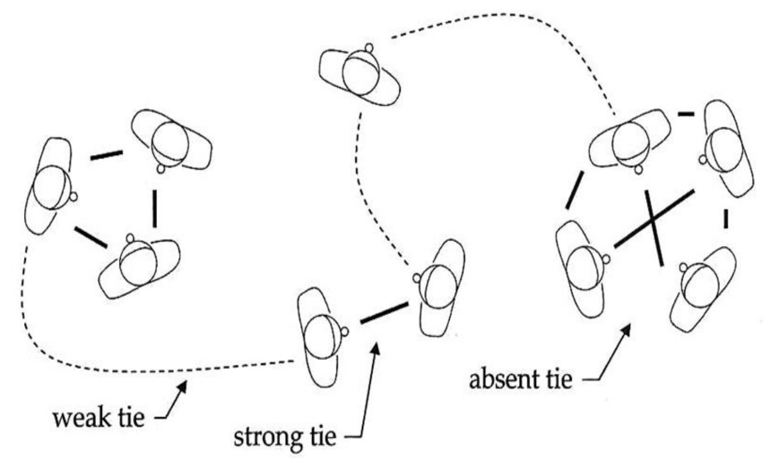
The outcomes of the Random Graphs put on society brings us to another model for the modulation of small worlds. Small World Graphs were developed almost 30 years after Milgram's famous experiment by the Australian physicist Duncan Watts and his mentor Steven Strogatz in 1998. The interest in the small world phenomenon got really big after their published study in the magazin "Nature".

Their model starts where the Random Graphs just stopped. Duncan asked himself:

"How big is the chance that my two best friends know each other?"

As described in a random graph the chances that my two best friends know each other are the same than a Chinese cook knows an American taxi driver. But we are all part of clusters with our friends where everybody knows everyone else. [24]

Fig. 1 Weak and Strong Ties [40]



The sociologist Mark Granovetter presumed that the network behind a clustered society consists of small fully connected circles of friends (strong ties) and the weak ties connecting them to their acquaintances. This is why weak ties are playing an important role in getting a new job or spreading rumors because they are connecting us to the outside world. [25]

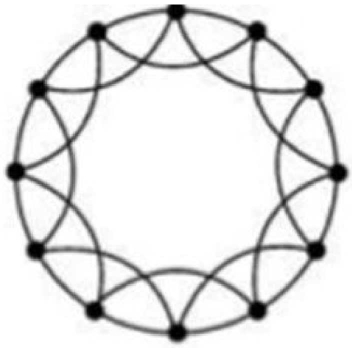
To proof the existence of clustering in social networks it needs to be possible to measure clustering. In order to be able to do this Watts and Strogatz introduced a quantity called the clustering coefficient. The clustering coefficient c for a node is given by the amount of links between the nodes within its neighborhood divided by the number of links that could possibly exist between them. [26]

Let's take a group of four good friends. If they all know each other it would be possible to connect them with six friendship links. It also might be that some of them are not good friends with each other. So in total they would only be connected by four friendship links. In this case the clustering coefficient would be $\frac{4}{6} \approx 0.66$. This gives an opportunity to measure how well connected a circle of friends is: the closer the clustering coefficient is to 1 the better the group of friends is connected.

Another example that proves clustering is present is society is the concept of the Erdős number. The mathematician Paul Erdős is mainly known for his countless theorems and proofs e.g. the Random Graphs. But he is also well known for the huge amount of his publications which are 1500 published papers with 507 coauthors. Because it's a great honour for scientist to have been working together with Erdős and to keep track of the different coautherships the Erdős number was introduced. Erdős himself has the number zero and someone who published a paper together with him would have the number one. Most mathematicians have a small number and are only about five steps away. [27]

The existence of the Erdős number already shows that scientist form a highly interconnected network in which all scientists are linked to each other over their co-authorships. The smallness of the average Erdős number indicates that it is a small world network. The unique feature that scientist publish their "social ties" regularly makes them easy to study. Barabasi and his team studied the publications of 70 975 mathematicians connected by 20 000 coauthorship links between 1991 and 1998. If they would have chosen their coauthors randomly the clustering coefficient would have been very small 10^{-5}. In a random network the clustering coefficient is the probability that two nodes are linked. But the clustering coefficient in the network of scientists is about 10 000 times larger. This shows that clustering is evident is the network formed by scientists and their coautherships.[28]

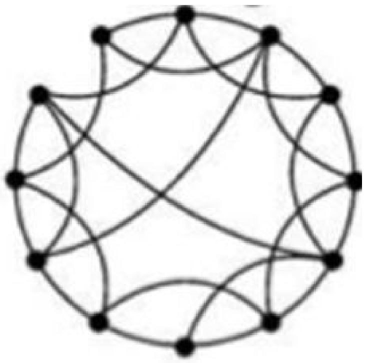
Fig.2 Regular Graph [41]



Duncan Watts and Steven Strogatz started from a circle of nodes to model networks with a high clustering coefficient. Every node is connected to its intermediate and next nearest neighbors. In this case the clustering coefficient would be $\frac{3}{6} = 0.5$. Compared to a random network, in this case of twelve nodes the the clustering coefficient would be 0.33, but for a billion nodes it would be $\frac{4}{1\,000\,000\,000}$.

Now it is a highly clustered model but the small world is gone e.g. to get from the top node to the node at the very bottom half of the circle needs to be visited. [29]

Fig.2 Small World Graph [42]



To give this model also the small world properties a few extra links were added. Because in reality we are all having friends who don't live next door. Consequently, a realistic model needs to allow distant links, as well. These stretched links offer shortcuts between the distant nodes and are drastically shortening the average separation between all the nodes. Moreover, they won't change the high clustering coefficient.

With these model Watts and Strogatz showed that huge networks don't need to be full of random links to have small world features. Just some randomly placed links to clusters far away will add those features.[30]

3.3 Scale Free Networks

Almost at the same time when Duncan Watts and Steven Strogatz published their study about Small World Graphs Albert-Laszlo Barabási made a huge discovery, as well. His research group was trying to better understand complex networks focussing on the World Wide Web. The results of their research were not combinable with either the Random Graphs by Erdös and Rényi or the Small World Graphs by Watts and Strogatz. Barabási and his team figured out after they analysed the WWW with their own crawler that the WWW consists out of a bunch of hubs. Hubs are nodes with an extraordinary large amount of links. But both previous model don't include nodes with significantly more links than the average node has. [31]

The Canadian journalist Malcolm Gladwell describes in his book "The tipping point" a simple test to measure how social someone is. He would give the person a list of 248 surnames and ask the person to give him/herself a point for every person he/she knows with that name. Multiple count, as well. If one of the names on the list would be e.g Daniel and the person knows three Daniels: He/She would get three points. In a group of mostly highly educated people the average was 39. But the actual surprise was the range of the test results. In a random group of people from different backgrounds the lowest score was 9 and the highest 118. And even in a uniform group of similar age, education and income the range was from 16 to 108. He came to the conclusion *"Sprinkled among every walk of life … are a handful of people with a truly extraordinary knack of making friends and acquaintances. They are connectors"* [32]

It is easy to relate to this statement when having a look at social networks like Facebook or Instagram. People who are "connectors" are a very important part of society when it comes to creating trends, fashion and bringing people together. The appearance of hubs is not only a phenomenon on the WWW it is also very present in society. The hubs are significant because they are creating short paths between any two nodes in the system and making it into a small world. E.g. on the WWW two randomly picked webpages are about 19 clicks away from each other whereas Google a giant hub is only one or two clicks away from any web page. [33]

To include the existence of hubs in modelling small worlds, Scale-Free Networks were introduced by Albert Laszlo Barabasi and Reka Albert in their article published in the “Science” paper in 1999.

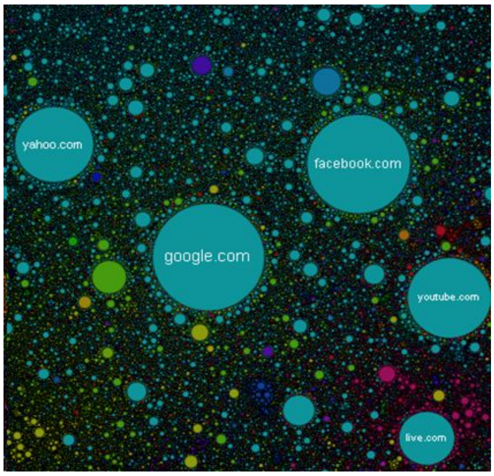


Fig. 3 Graphic of the five big hubs in the WWW [43]

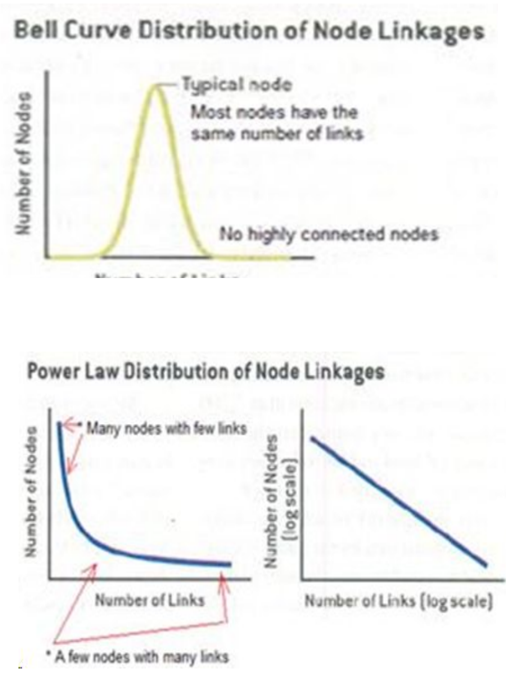


Fig. 4 & 5 Degree Distributions [44]

For a better comparison: In Random Networks the degree distribution follows a bell curve which means that most of the nodes have the same amount of links and no nodes with very many links exist. An example for a Random Network would be the national highway network. Most cities (nodes) are roughly served by the same amount of highways (links).

Whereas in a Scale-Free Network the degree distribution follows a power law. This is represented by many nodes with only a few links and a few nodes (hubs) with a large amount of links. This is comparable to the air traffic system in the United States. There are a lot of little airports and a few very big ones who connect the small ones with each other. [34]

Barabási and Albert figured out that the appearance of the hubs follows strict mathematical laws and can be described with a power law distribution [35]. A power law is a special mathematical relationship between two quantities in which one quantity varies as a power of the other. In scale free networks the degree distribution defined as the probability that a randomly chosen node has k connections, can be expressed as: $P_{deg}(k) \propto k^{-\gamma}$. γ is some exponent and this form of $P_{deg}(k)$ decays slowly as the degree k increases, increasing the probability of finding a node with a huge amount of links. [36]

Furthermore, they revealed that real networks like society or the WWW follow two laws:

- Growth: Each network starts with one first node and than grows with the addition of new nodes
- Preferential attachment: These newly added nodes, when deciding where to link, prefer the nodes that already have more links.

This is another very important difference to previous models because they assumed a fixed amount of links and don't see the dynamic character of the network. This leads to a simple algorithm which got developed by Barabási and Albert to demonstrate how a scale free networks grows:

- A: For each period of time a new node is added to the network
- B: It's assumed that each node connects to the existing nodes with two links. The probability that it will choose a node is proportional to the number of links the chosen node has.

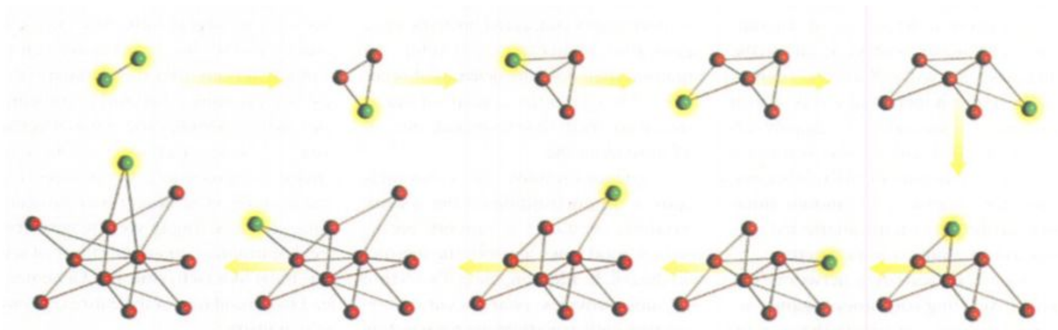


Fig. 6 Illustration of the algorithm described above [45]

This implies that if a node has twice as many links than another node, it is also twice as likely that the new node will connect to the better connected one. This will lead to the evolution of the hubs. [37]

Examples of real networks which can be modeled by Scale-Free Networks are:

Network	***Nodes***	***Links***
Cellular metabolism	Molecules involved in burning food for energy	Participation in the same biochemical reaction
Hollywood	Actors	Appearance in the same movies
Internet	Routers	Optical and other physical connections
Protein regulatory network	Proteins that help to regulate a cell's activities	Interactions among proteins
Research collaborations	Scientists	Co-authorship of papers
Sexual relationships	People	Sexual Contact
World Wide Web	Web pages	URLs

[38]

4 Conclusion

In summary, the evolution from network science included the Random Graphs by Erdös and Rényi which were redefined by Watts and Strogatz with the development of the Small World Graphs including clustering in the model. Albert-Laszlo Barabási and Reka Albert changed the view of networks as a static system to a dynamic system which follows the laws growth and preferential attachment. The model of Scale-Free Networks which were introduced by them also allows nodes with an significantly large amount of links. These hubs are very important in creating short paths in the total system.

The cited examples demonstrate the importance and potential of the network science in our modern society, cause we are all in daily contact with these networks. It helps us to understand how they work and how we can prevent them from damage. It is stunning to see that all of these networks can be understand and described with the same mathematical concepts.

The different research results are used in companies to place their webpages as efficient as possible in the WWW and helps marketing experts to understand how a new trend spans, as well, as how business relationships are chosen. Moreover scientist use the results of network science to understand how a virus like HIV can spread in society and how it can be prevented as well as to understand how the molecular metabolism in our body works. [39]

Index

[1] Stanley Milgram. The Small World Problem. Psychology Today, 1967: p.60-67
[2] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.27
[3] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.28-29
[4] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.29
[5] 198,7 million (1967) United States Census Bureau, Weltbank
[6] Olle Häggström. Streifzüge durch die Wahrscheinlichkeitstheorie. Springer, 2006: p.135
[7] Judith S. Kleinfield. Six Degrees of Separation: An Urban Myth. Psychology Today, 2002
[8]https://searchengineland.com/googles-search-indexes-hits-130-trillion-pages-documents-263378
[9] H.Jeong, R.Albert, and A.-L.Barabási. Internet: Diameter of the world-wide-web. Nature, 401:130-131, 1999.
[10] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.32
[11] H.Jeong, R.Albert, and A.-L.Barabási. Internet: Diameter of the world-wide-web. Nature, 401:130-131, 1999.
[12] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.33
[13] H.Jeong, R.Albert, and A.-L.Barabási. Internet: Diameter of the world-wide-web. Nature, 401:130-131, 1999.
[14] H.Jeong, R.Albert, and A.-L.Barabási. Internet: Diameter of the world-wide-web. Nature, 401:130-131, 1999.
[15] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.34
[16] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.34
[17] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.35
[18] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.39-40
[19] Olle Häggström. Streifzüge durch die Wahrscheinlichkeitstheorie. Springer, 2006: p.146
[20] Olle Häggström. Streifzüge durch die Wahrscheinlichkeitstheorie. Springer, 2006: p.147
[21] Olle Häggström. Streifzüge durch die Wahrscheinlichkeitstheorie. Springer, 2006: p.147
[22] Olle Häggström. Streifzüge durch die Wahrscheinlichkeitstheorie. Springer, 2006: p.151

[23] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.44
[24] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.45
[25] Mark S. Granovetter. The Strength of Weak Ties. American Journal of Sociology 78, 1973: 1360-1380.
[26] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.47
[27] https://www.oakland.edu/enp/ maintained by Jerrold W. Grossman
[28] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.47-49
[29] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.51-53
[30] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.51-53
[31] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.54
[32] Malcolm Gladwell. The Tipping Point: How Little Things Can Make a Big Difference. Little Brown, New York, 2002: p.41
[33] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.58
[34] A.L.Barabási, Eric Bonabeau. Scale-Free Networks. Scientific American, 2003
[35] A.L.Barabási, Eric Bonabeau. Scale-Free Networks. Scientific American, 2003
[36] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.68
[37] A.L.Barabási. Linked: How everything is connected to everything else and what it means for business, science, and everyday life. Basic Books, New York, 2014: p.86-87
[38] A.L.Barabási, Eric Bonabeau. Scale-Free Networks. Scientific American, 2003
[39] A.L.Barabási, Eric Bonabeau. Scale-Free Networks. Scientific American, 2003

Figures:
[40]https://www.leadershipcloseup.com/2012/12/14/strength-of-weak-ties-in-social-networking-seek-to-be-worth-knowing/
[41]https://slideplayer.com/slide/6262377/21/images/22/D.+J.+Watts+and+Steven+Strogatz+%28June+1998%29.jpg
[42]https://slideplayer.com/slide/6262377/21/images/22/D.+J.+Watts+and+Steven+Strogatz+%28June+1998%29.jpg
[43]https://tekeye.uk/computing/how-many-websites-are-there
[44]https://slideplayer.com/slide/9359https://slideplayer.com/slide/9359558/558/28/images/2/Random+Networks+Vs.+Scale+Free+Networks.jpg
[45] A.L.Barabási, Eric Bonabeau. Scale-Free Networks. Scientific American, 2003