On Dividing a Square Into Triangles

Seminar talk about “Elegant Proofs”

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FH Wedel

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Square and triangles

First approach

Dissect a square into triangles?

Square and triangles

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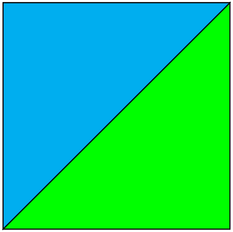

Figure: Dissection with an even count

Square and an odd number of triangles

Another approach

Dissect a square into an odd number of triangles?

Square and an odd number of triangles

Another approach

Dissect a square into an odd number of triangles?

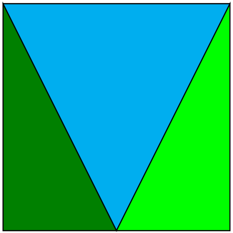

Figure: Dissection with an odd count

Triangles with equal area

Triangles with equal area

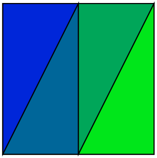

Figure: Even number

Triangles with equal area

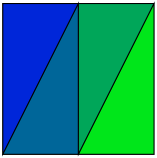

Figure: Even number

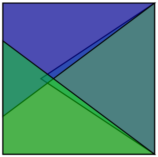

Figure: Odd number

Question

"Can a square $\mathbf{S}$ *be divided into an odd number of nonoverlapping triangles* $\mathbf{T_i}$*, all of the same area?"*

Source: [5]

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A solution?

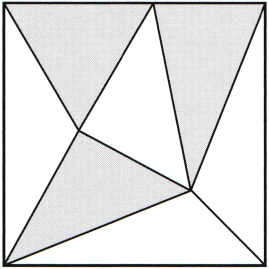


Figure: Dissection with nearly equal sized triangles

Image source: [1]

Monsky's Theorem

Theorem (Monsky[1])

It is not possible to dissect a square into an odd number of nonoverlapping triangles, all of same area.

[1] Paul Monskey: Author of the proof

Profile

- given in “On Dividing a Square into Triangles” [4]
- by Paul Monskey
- in 1970

Profile

- given in “On Dividing a Square into Triangles” [4]
- by Paul Monskey
- in 1970

- uses combinatorics
- and an area of algebra: Valuation theory
  (in German: Bewertungstheorie)

Steps

Steps

Square coloring

Steps

Square coloring

↓

Determinant lemma

Steps

Square coloring

↓

Determinant lemma

↙

Rainbow triangle area

Steps

Square coloring

↓

Determinant lemma

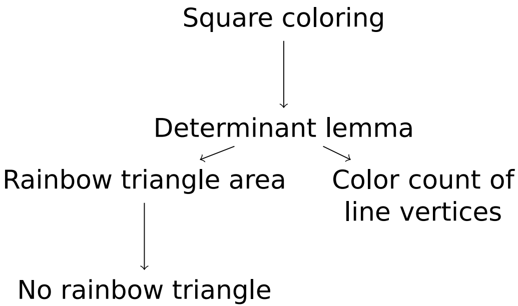
↙

Rainbow triangle area

↓

No rainbow triangle

Steps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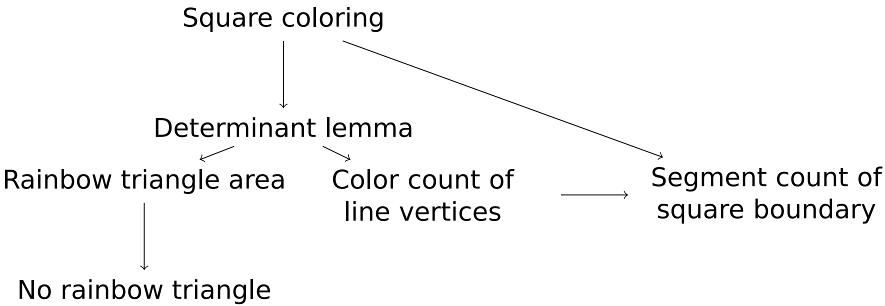
Determinant lemma
Rainbow triangle area
Color count of line vertices
No rainbow triangle

Steps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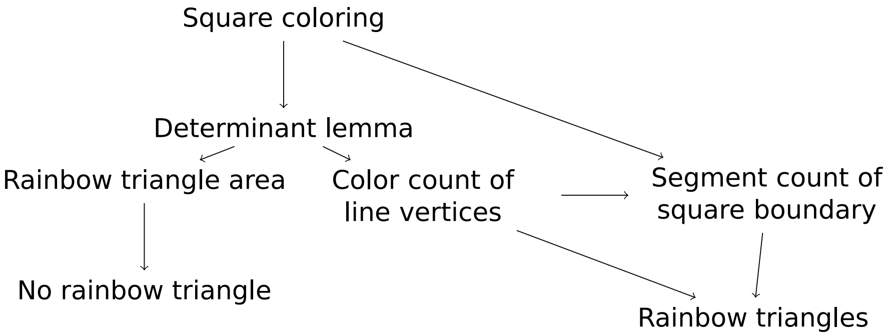
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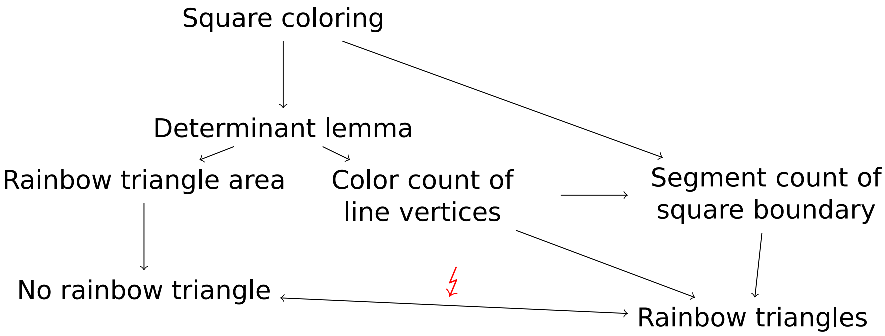
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Valuation theory

Definition (Valuation)

$$\upsilon : F \to V \cup \{\infty\}$$

where F is a field and V an additive ordered group and

$$\upsilon(a) = \infty \Leftrightarrow a = 0$$
$$\upsilon(a \cdot b) = \upsilon(a) + \upsilon(b)$$
$$\upsilon(a + b) \geq \min(\upsilon(a), \upsilon(b))$$

Further informations in [3].

Special valuation

Definition (absolute value)

$$|\,| : \mathbb{R} \to \mathbb{R}_{\geq 0}$$

This function has the properties:

$$|x| = 0 \Leftrightarrow x = 0$$
$$|x \cdot y| = |x| \cdot |y|$$
$$|x + y| \leq |x| + |y| \quad \text{(triangle inequality)}$$

Special valuation

Definition (absolute value)

$$|\,| : \mathbb{R} \to \mathbb{R}_{\geq 0}$$

This function can have the properties:

$$|x| = 0 \Leftrightarrow x = 0$$

$$|x \cdot y| = |x| \cdot |y|$$

$$|x + y| \leq \max\{|x|, |y|\} \quad \text{(non-Archimedean)}$$

Special properties of absolute values

- $|1| = 1$
- $|-1| = 1$
- $|-x| = |x|$

Example of an absolute value

Definition (p-adic value)

$$|r|_p := p^{-k}, \quad |0|_p = 0$$

with

- p is a prime
- $k \in \mathbb{Z}$
- $p^k \cdot \frac{a}{b} = r$

for a given p and $r \in \mathbb{Q}$

Examples for p-adic values

Example

- $|2|_2 = \left|2^1 \cdot \frac{1}{1}\right|_2 = 2^{-1} = \frac{1}{2}$
- $\left|\frac{3}{4}\right|_2 = \left|2^{-2} \cdot \frac{3}{1}\right|_2 = 2^{-(-2)} = 4$
- $\left|\frac{1}{3}\right|_2 = \left|2^{0} \cdot \frac{1}{3}\right|_2 = 2^{-0} = 1$
- $\left|\frac{6}{7}\right|_2 = \left|2^1 \cdot \frac{3}{7}\right|_2 = 2^{-1} = \frac{1}{2}$

Coloring

We need an non-Archimedean absolute value υ in which $\upsilon(2) < 1$.

Definition (Coloring)

$$(x, y) \text{ is colored } \begin{cases} \text{blue} & \text{if } \upsilon(x) \geq \upsilon(y) \wedge \upsilon(x) \geq \upsilon(1) \\ \text{green} & \text{if } \upsilon(x) < \upsilon(y) \wedge \upsilon(y) \geq \upsilon(1) \\ \text{red} & \text{if } \upsilon(x) < \upsilon(1) \wedge \upsilon(y) < \upsilon(1) \end{cases}$$

Coloring example

Example (from [1])

- 2-adic value
- fractions of the form $\frac{k}{20}$

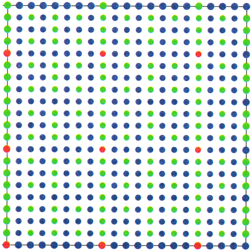


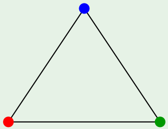
Figure: Example for a colored square

Rainbow triangle

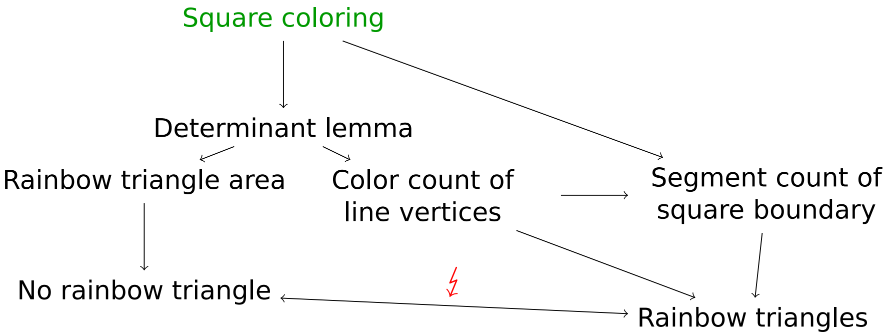
Definition (rainbow triangle)

A triangle generated by a red, green and blue vertex.

Example



Steps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Determinant lemma

Definition

$$\mathbf{M} := \begin{pmatrix} x_b & y_b & 1 \\ x_g & y_g & 1 \\ x_r & y_r & 1 \end{pmatrix}$$

Theorem

For any blue point (x_b, y_b)*, green point* (x_g, y_g) *and red point* (x_r, y_r) *the value of the determinant of* $\mathbf{M}$ *is at least* 1*:*

$$\upsilon(\det(\mathbf{M})) \geq 1$$

Proof for determinant property

Proof.

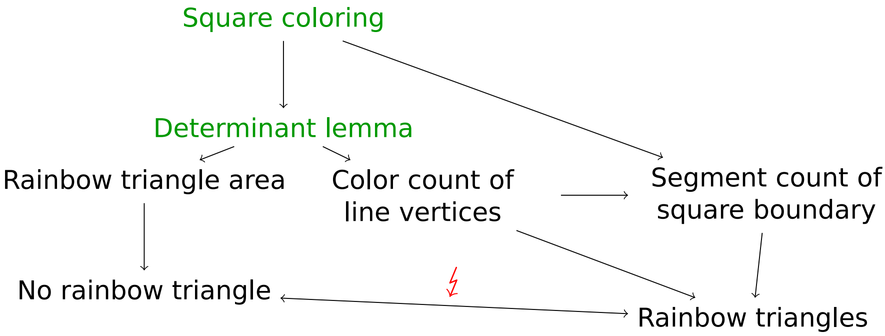
$$\upsilon(\det(\mathbf{M})) = \upsilon(x_b \cdot y_g) = \upsilon(x_b) \cdot \upsilon(y_g) \geq \upsilon(1) \cdot \upsilon(1) = 1$$

□

Hint

Follows from the coloring on frame 15.

Steps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Area of a rainbow triangle

Target

The area of a triangle of a dissection must be $\frac{1}{n}$. A rainbow triangle should violate this, so it must not be a part of a dissection.

Theorem

The area of a rainbow triangle cannot be zero or $\frac{1}{n}$ for odd n.

Area is not zero

Hint

The area of an triangle can be computed with a determinant (see for example [2]):

$$A = \frac{1}{2} \cdot \left| \det \begin{pmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{pmatrix} \right| = \frac{1}{2} \cdot \left| \det(\mathbf{M}) \right|$$

Area is not zero

Hint

The area of an triangle can be computed with a determinant (see for example [2]):

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Proof.

The value of the determinant is at least one (see 19). The half cannot be zero. □

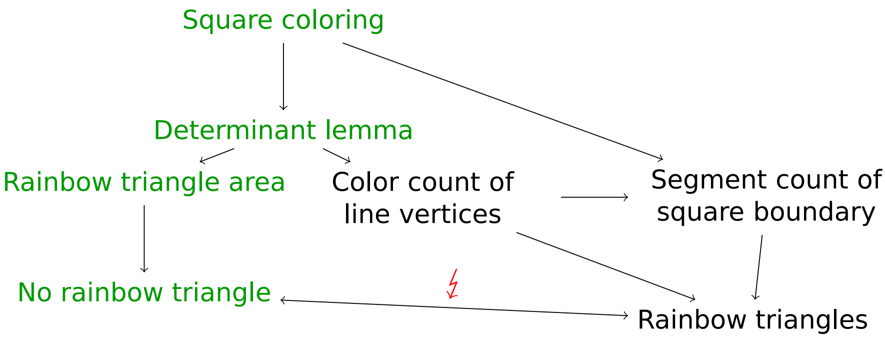
Area $\neq \frac{1}{n}$ for odd n

Proof.

1. Suppose the area A of a rainbow triangle is $\frac{1}{n}$.
2. The value of the determinant (see 19) is at least one.
3. But if the n is odd, it should be less then one. $\lightning$

$\square$

Steps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Existence of rainbow triangles

Target

We showed already, that there must not be a rainbow triangle. Now we show, that there must be at least one.

Theorem

Suppose that no face contains vertices of all three types and that R has an odd number of faces of type $\alpha\beta$. Then some T_i has vertices of all three types.

Source: [4]

Existence of rainbow triangles

Target

We showed already, that there must not be a rainbow triangle. Now we show, that there must be at least one.

Theorem

Suppose that no line contains vertices of all three types and that the square has an odd number of boundary lines of type red-blue. Then some triangles has vertices of all three types.

No lines with all three colors

Theorem

Any line of the plane receives at most two different colors.

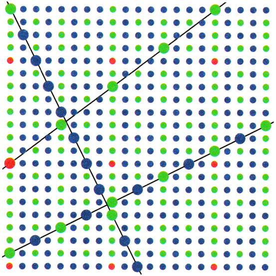


Figure: Example for a square coloring (with lines)

Image source: [1]

No lines with all three colors

Theorem

Any line of the plane receives at most two different colors.

Proof.

1. Suppose there lie red, green and blue points on a line.
2. The value of the determinant (see 19) would be zero.
3. This is a contradiction, because it must be at least one. $\lightning$

$\square$

red-blue segments

Definition (red-blue segment)

A segment is called a red-blue segment if one endpoint is red and the other is blue.

Theorem

A concatenation of segments starting with a red vertex and stopping with a blue vertex contains an odd number of red-blue-segments.

Segment count of square boundary

Theorem

The boundary of the square contains an odd number of red-blue segements.

Segment count of square boundary

Theorem

The boundary of the square contains an odd number of red-blue segements.

Hint

Follows from coloring (see 15) and $\nu(0) = 0$, $\nu(1) = 1$ for all valid valuations.

Proof.

$$\begin{array}{ccc} (0,1) & - & (1,1) \\ | & & | \\ (0,0) & - & (1,0) \end{array}$$

□

A dissection contains rainbow triangles

Theorem

- *Triangles with at most two colors at its vertices contains an even number of red-blue segments*
- *Rainbow triangles contains an odd number of red-blue segments*

Proof.

There must be an odd number of rainbow triangles, because the square boundary contains an odd number of red-blue segments. □

Existence of rainbow triangles

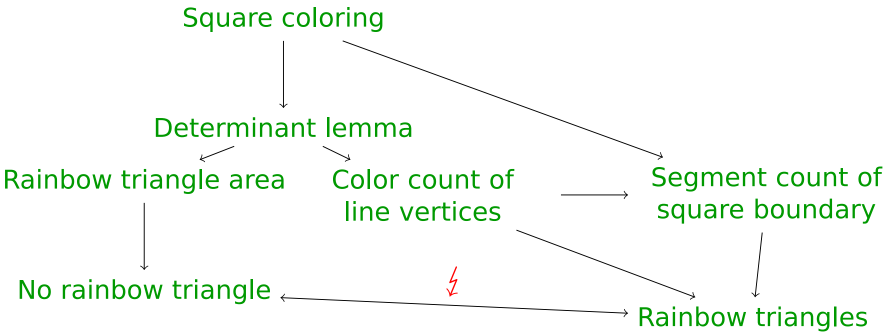
Target

We showed already, that there must not be a rainbow triangle. Now we show, that there must be at least one.

Theorem

Suppose that no line contains vertices of all three types and that the square has an odd number of boundary lines of type red-blue. Then some triangles has vertices of all three types.

Steps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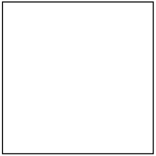
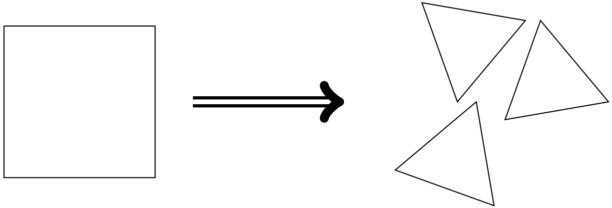
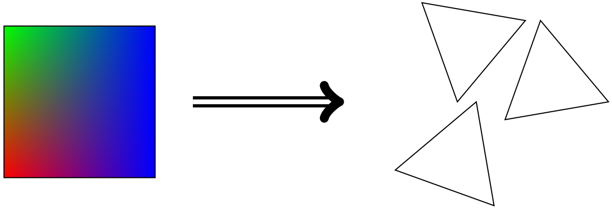
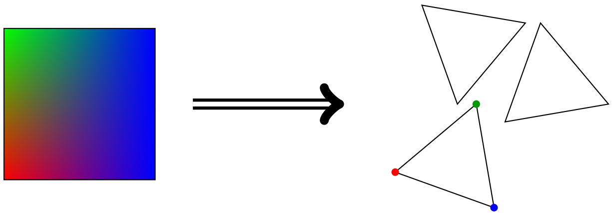
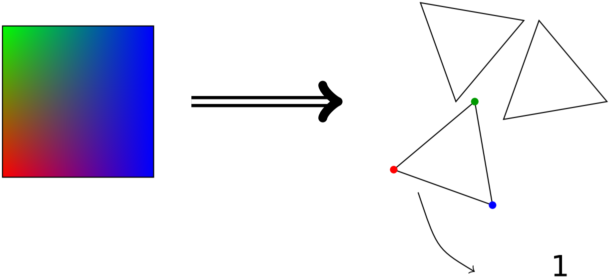


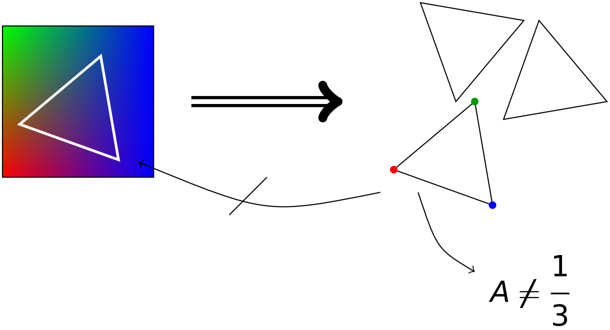








$$A \neq \frac{1}{3}$$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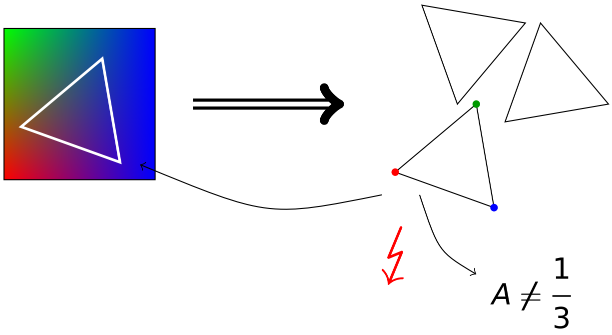
$A \neq \frac{1}{3}$

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References

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- Gerd Fischer. *Lineare Algebra. Eine Einführung für Studienanfänger*. 18th ed. 2014.
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- Fred Richman and John Thomas. "Problem 5479". In: *The American Mathematical Monthly* 74 (1967).

Quod erat demonstrandum

Thank you for your attention.

Are there any questions left?