FIVE COLORING PLANE GRAPHS

Seminar WS2016

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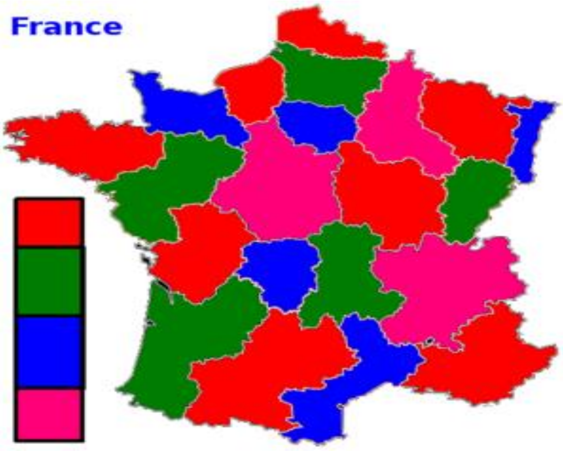
Content

Introduction

Graph coloring is one of the most well-known subjects in the field of graph theory which has many applications like making schedule or time tables, communication networks and coding theory genomics and map coloring which in this paper we will discuss the map coloring.[3]

The problem

Let's start with the explanation of the problem, the problem we want to solve is that we want to color a map such that if two regions are neighbors they receive different colors and easy way to solve this problem is to give each region a unique color but this is not so clever, the interesting part of this problem is to color this map such it uses as few colors as possible for instance this is the map of France which is colored only by four colors and the color of each region is different from its neighbors.

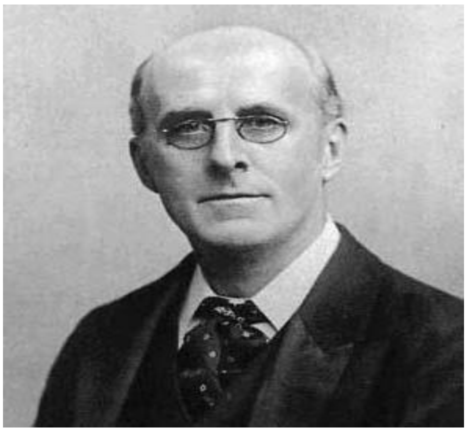


History

The history of map coloring theorem goes back to 1879 when an English mathematician Alfred Kempe used a method called Kempe chain to prove the four coloring theory, however the way he proved this theory was not completely correct but still it was so important for the history of map coloring theory.

After 11 years Percy Heawood found an error in kempe's method, he found a counterexample map which was not colorable by four colors according to kempe's method and of course we know that in mathematics if someone claims something and we want to prove them wrong we need a counterexample. After founding then error Heawood used some of Kempe's ideas and proved five color theorem.

Eventually in 1976 Apple and Haken announced that four color would suffice which the proof was computer based.

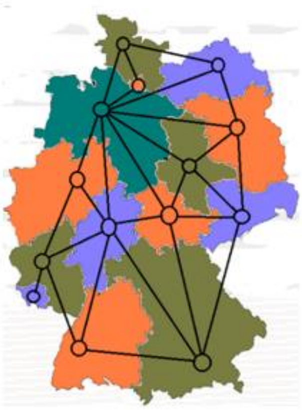


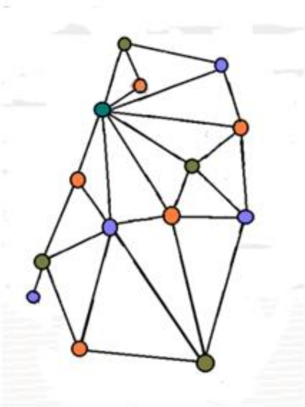
Alfred Kempe

Percy Heawood

Translating map coloring to graph coloring

Now if we take a closer look to map coloring theorem we see that we can translate it to graph theory if we convert each region to a vertex and whenever there is a border between two regions we connect the vertices by an edge. Now we want to color a graph such that if two vertices are connected by an edge they receive different colors.

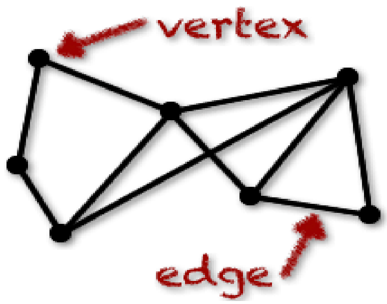




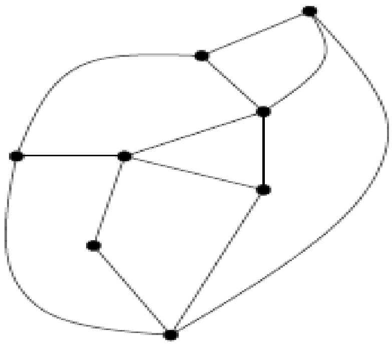
Some definitions from graph theory

Before we start with the proofs let's review some terms of graph theory.

A graph is a collection of vertices (the points) which are connected by the edges (the lines). Some graphs can be directed and some can have no edges at all (forest). And there could be more than one edge coming out of one vertex.



For example the graph below has 8 vertices, 13 edges and 7 regions and of course when we want to count the regions we also count the outer one.

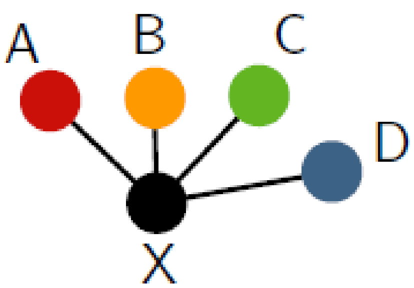


And for a connected graph we can apply the Euler's formula which always equals two.

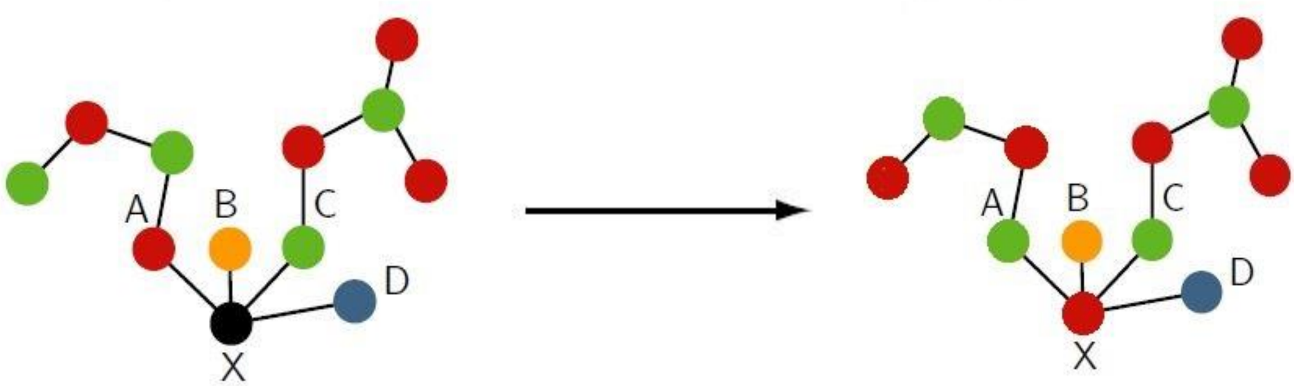
$$|V|-|E|+|R|=2$$

Kempe's chain method

Kempe used an approach called kempe's chain to proof four color theory. Consider If we have a map that every region is colored: A=red, B=yellow, green=C and D=blue, and now we want to color x and it is surrounded by all four regions *and now we want to color x in such a way that we still use four colors*. Hence if *X* is surrounded by regions *A*, *B*, *C*, *D* then, there are two cases to consider:

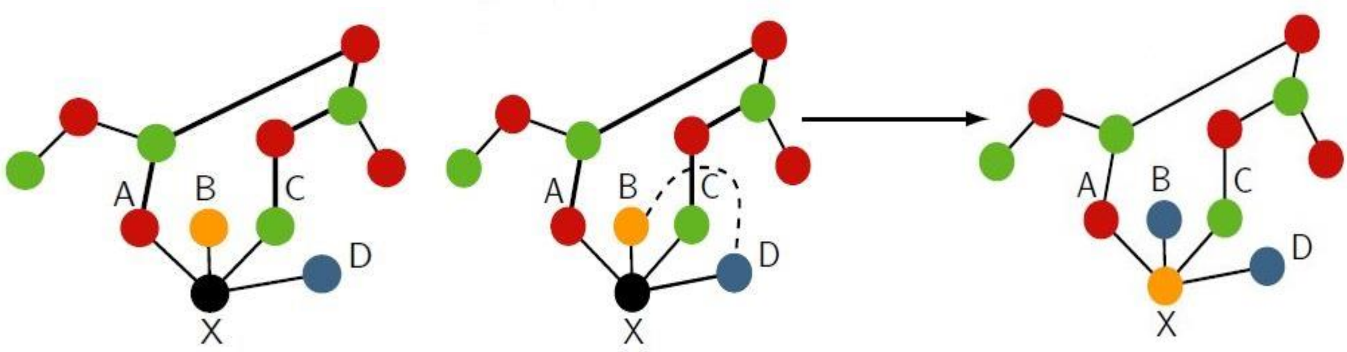


1. There is no chain of adjacent regions from *A* to *C* alternately colored red and green:



If this case holds Change *A* to green, and then interchange the color of the red and green regions in the chain joining *A*. Now C gets the color green hence, we can color X with the color red.

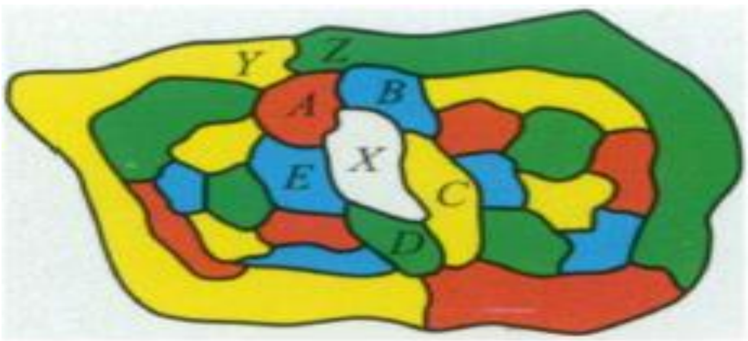
2. There is a chain of adjacent regions from *A* to *C* alternately colored red and green.



If this case holds we know that there is no yellow- blue chain in the graph and when we don't have a chain case one holds and we could change the color of B from yellow to blue and give yellow to X.

Kempe's chain counter-example

Still the However Kempe's proof has considered very important to the future of map coloring theorem but the way he proved this theory was not correct and Heawood found an error in this proof, He provided a map which was not colorable by kempe's chain method.



In this map the uncolored region x is surrounded by five regions A,B,C,D and E colored as red, blue, yellow, green and blue. According to kempe's chain we check if we have to follow case one or case two, since we have red-green chain between A and D (case two) and we would change the colors in B or C or E in order to give X a different color from its neighbors which is not possible.

Let's take a closer look, according to kempe's method we would give the color of C to E, now E has the color yellow and C has the color yellow too and we can give the color blue to X, we expect that the problem is solved but now we see that B is a problem here. So we come to a conclusion that kempe's chain method won't work always.

6-colorable plane graph

Now let' prove that every plane graph is six colorable.

Theory:

Any map can be colored with six or fewer colors in such a way that no adjacent vertices receive the same color.

Lemma:

Let G be any simple plane graph with n > 2 vertices. Then G has a vertex of degree at most 5.

Proof of lemma:

Let's proof this lemma by contradiction. We assume that we have a planar graph named G and all the vertices of this graph have degree six or more than six and we already know that some of the degree of a planar graph is twice the number of edges of the graph. As we said in this case every vertex has degree of six or more so we can write

$6v \leq 2e$

$3v \leq e$ (1)

Since G is a planar graph we can apply the Euler's formula for the graph G:

$|V| - |E| + |R| = 2$ (Euler's formula)

$e \leq 3v - 6$ (2)

From (1) and (2) we have:

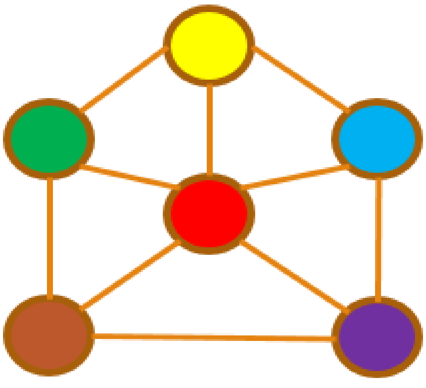
$3v \leq e \leq 3v - 6$ (3)

Since $3v > 3v - 6$ the inequality (3) can't be possible. Hence, we come to a contradiction to our earlier assumption that all vertices of G have degree six or more. Hence, we can say if we a planar graph G, then G has a vertex of degree at most 5.

Base case:

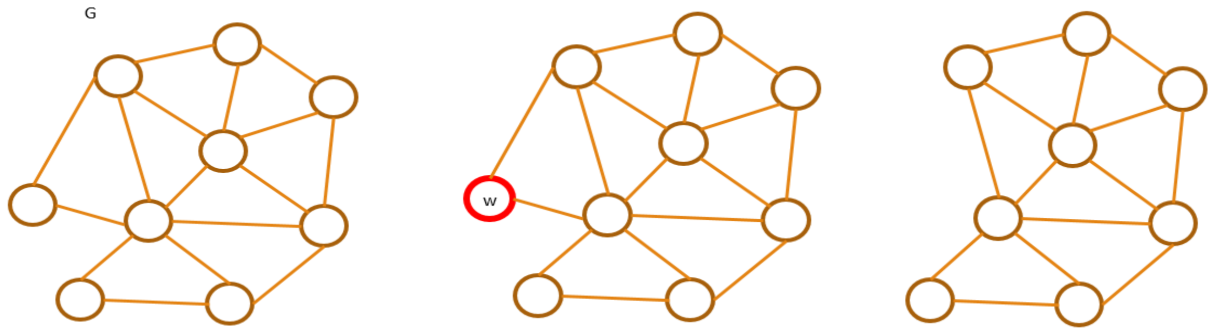
If we say v is the number of vertices, for small values of n this is obvious (**v≤6**).

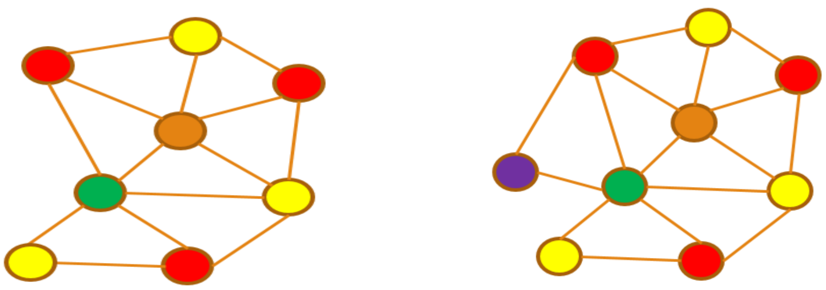
We give each vertex a unique color and problem is solved.



Proof for v=n+1:

Let G be any simple planar graph on v = n+ 1 vertices. From our lemma above, we know that G must have some vertex w of degree ≤ 5. Remove w from G. now has v = n vertices and we may apply our base case to know it can be properly colored in 6 colors. Now, we can think of this as coloring all of G except for w that we took away. But, since w has degree at most 5, one of the 6 colors will not be used for any of the neighbors of w and we can finish coloring G.





All planar graphs G can be 5-list colored:

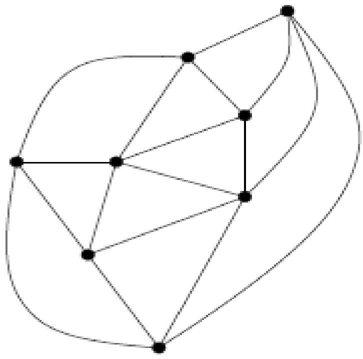
Theory:

All planar graphs G can be 5-list colored (we need only five colors to color a plane graph).

Proof:

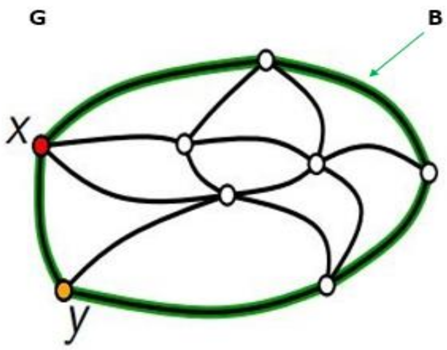
The list chromatic of plane graphs means the smallest possible value to obtain a k-coloring for a plane graph G and each vertex can be colored only with a list of allowed colors.

Consider the graph G which is connected and that all the bounded faces of an embedding have triangles as boundaries. We call such a graph near-triangulated.



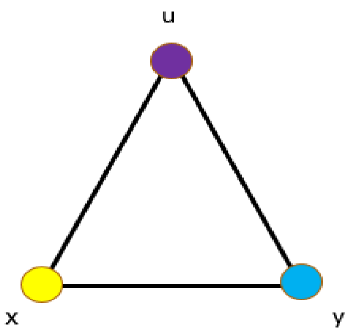
A near-triangulated plane graph

We can proof this theory by showing the following stronger statement for a graph G with V number of vertexes and E number of edges which is near-triangulated and B is the cycle bounding the outer region of graph, we make the following assumptions on the color sets $C(v), v \in V$:
(1) Two adjacent vertices x, y of B are already pre-colored.
(2) $|C(v)| \geq 3$ for all other vertices v of B.
(3) $|C(v)| \geq 5$ for all vertices v in the interior.



Base case:

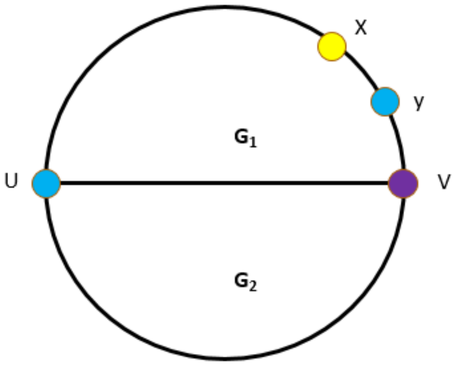
For number of vertices v=3 this is obvious. We have two pre colored vertices x and y which are colored yellow and blue and for coloring of vertex u we still have three options left of five colors which we gave it the color purple.

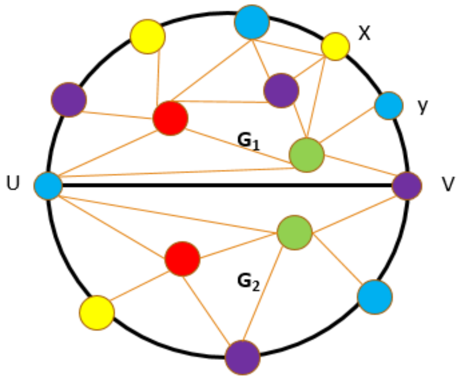


Now that we want proceed by induction for number of edges v>3 we face two cases:

Case 1:

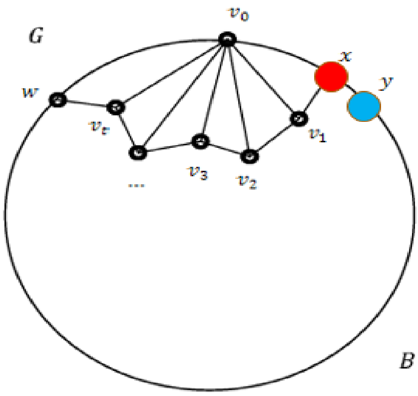
B has a chord which divides the graph G into two sub graphs G1 and G2. G1 contains x, y, u and v and it's near-triangulated and therefore has a 5-list coloring by induction. In this graph x and y are pre colored with the colors yellow and blue so we can color u and v with colors blue and purple. Now we look at the bottom part G2 for this graph we have the two pre colored vertices u and v now we see that the induction hypotheses also works for G2. Hence G2 can be 5-list colored. Since the graph G is made of sub graphs G1 and G2 we could say G can also be 5-list colored.



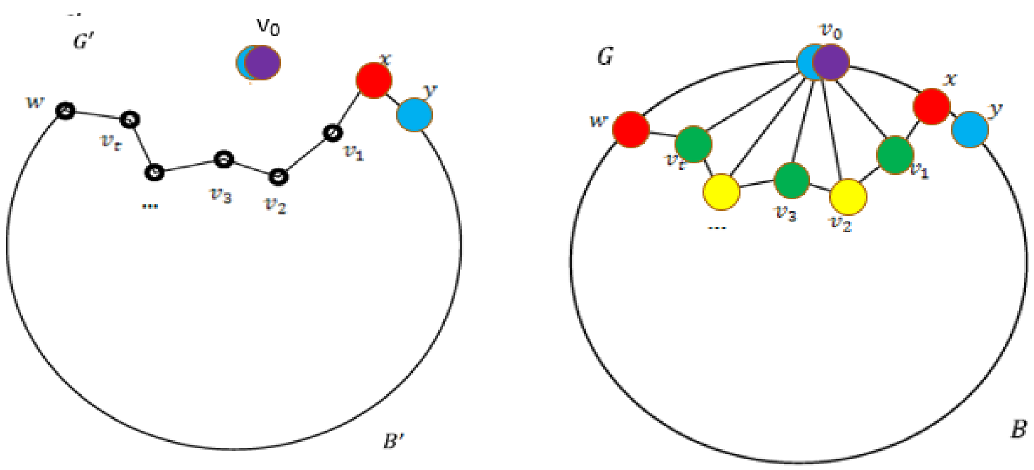


Case 2:

B has no chord. We have the vertices x and y pre colored now let v_0 be the vertex on the adjacent to the red-colored vertex x on B and Let $x, v_1, v_2, \dots, v_t, w$ be the neighboring vertices of v_0 and we know that G is near triangulated.

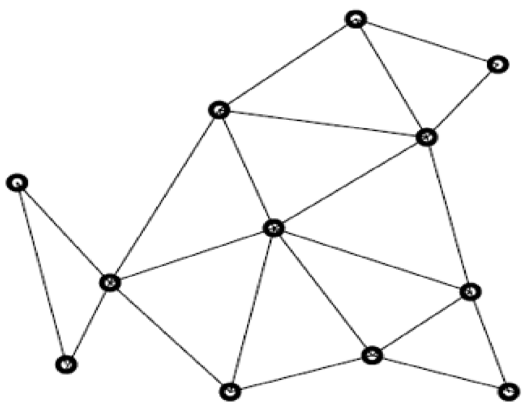


Now Construct the near-triangulated graph G' there must exist two colors purple and blue in $C(v_0)$ that are different from red. Remove the colors purple and blue from the color sets for each of the vertices $v_1, \dots, v_t$ since we know that v_0 will have one of those two colors. New color sets will be of the form $C(v_i) / \{\text{purple}, \text{blue}\}$ .If we take a look G' covers all of our assumptions so we can say it's five colorable by induction. Therefore we can extend the list coloring of G' to all of G.

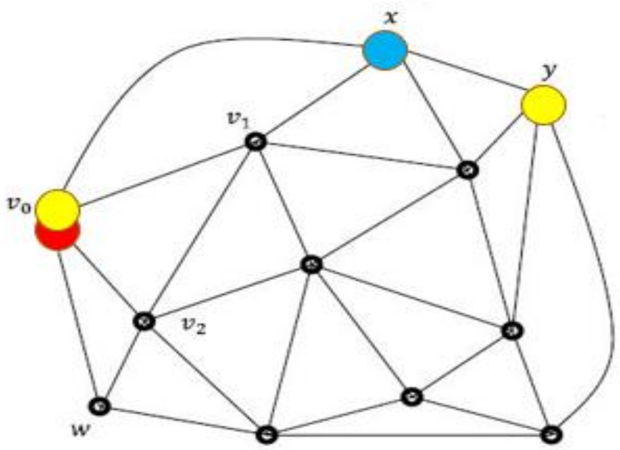


Example

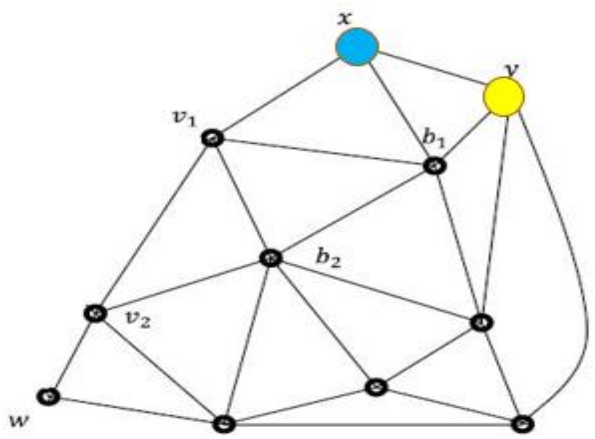
Now let's do an example for better understanding. We want to color the graph below using this colors {blue, yellow, red, green, purple}.



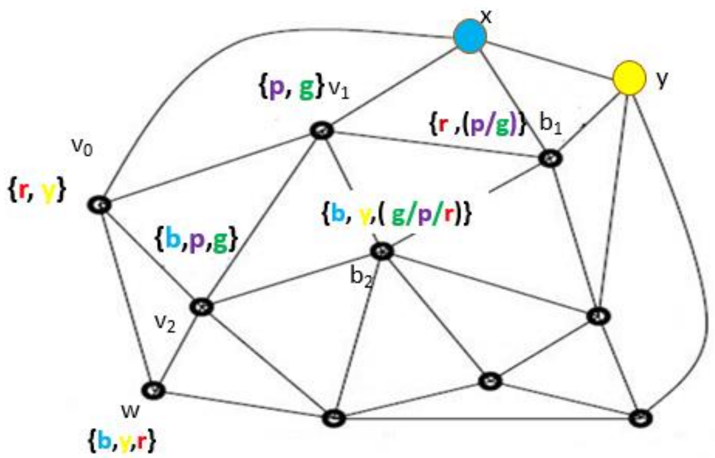
The graph is connected and it's triangulated for making it near triangulated we add an outer region for the graph and the we look if we are in case one or case two, since the outer region has no chord we are in case two. We start by naming two adjacent vertices of the graph x and y and we pre color them with the colors blue and yellow then name the other boundary vertex adjacent to x as v_0 , then we can label the interior vertices connected tov_0 as v_1, v_2 and the adjacent boundary vertex as w. We allow v_0 have the possibility of being either color red or yellow.



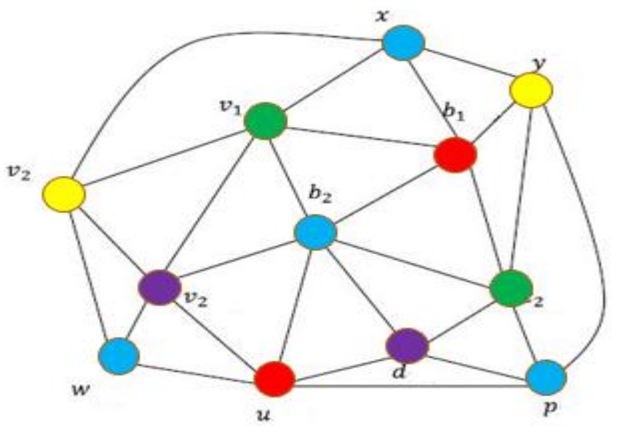
Now remove the vertex v_0 from our graph and all the edges connected to it. Now we know that the graph below can be colored with five colors according to our proof.



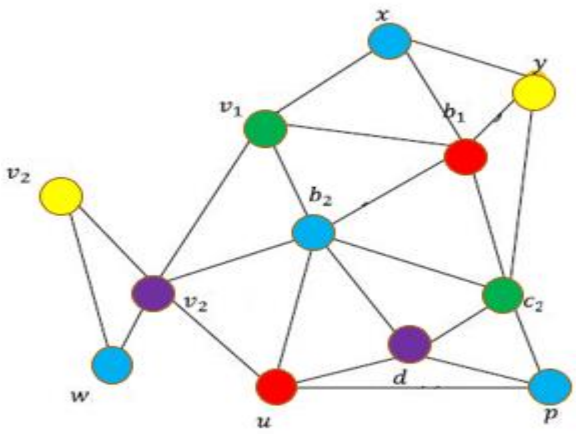
And in the graph below it is showed that how we can give colors to each node.



And if we choose our final color from the choices above we will have:



And finally we take back the extra edges that we added in the beginning to make our graph near triangulated.



Summary

As a summary in the end let’s look at what we have learnt from this topic.

The problem is how to color a map such that each region has a unique color to its neighbors using minimum number of colors. For solving this problem we can convert map coloring theorem to graph coloring theory.

Some mathematicians did proofs in this field for example Kempe publish his proof on four color theorem called kempe’s chain which his method was not correct and then later on Heawood found an error in kempe’s method and proved five color theorem.

Now we know how kempe’s chain method works and can prove that it’s not always working.

As a warm up we proved the six color theorem.

We proved the five color theorem and solved an example for better understanding.

With this we proved that every graph is six-colorable and five-colorable. There is another prove, which was not part of this seminar-topic, that shows every graph is even four-colorable but not less than four colors.

The minimum number of colors we need to solve our problem is at least four.

Reference

- [1] Proofs from THE BOOK-4th by Martin Aigner and Günter M. Ziegler.
- [2] Chromatic Graph theory by Gary Chartrand, Ping Zhang
- [3] https://en.wikipedia.org/wiki/Graph_coloring